Preprint typeset in JHEP style - HYPER VERSION

hep-th/0605225

CPHT-RR017.0306, ROM2F/2006/10 LPT-ORSAY-06-34, CERN-PH-TH/2006-084

Anomalies, Anomalous U(1)'s and generalized Chern-Simons terms

P. Anastasopoulos[1], **M. Bianchi** [2], **E. Dudas** [3,4,5], **E. Kiritsis** [3,6]

[1] *Department of Physics, University of Athens,*
Panepistimiopolis, Zografou- 157 84 Athens, GREECE

[2] *Dipartimento di Fisica, Università di Roma "Tor Vergata"*
I.N.F.N. - Sezione di Roma "Tor Vergata"
Via della Ricerca Scientifica, 1 - 00133 Roma, ITALY

[3] *CPHT, UMR du CNRS 7644, Ecole Polytechnique,*
91128 Palaiseau, FRANCE

[4] *LPT, UMR du CNRS 8627, Bat. 210, Univ. Paris-Sud,*
91405 Orsay Cedex, FRANCE

[5] *CERN Theory Division, CH-1211, Geneva 23, SWITZERLAND*

[6] *Department of Physics, University of Crete, 71003 Heraklion, GREECE*

Abstract: A detailed analysis of anomalous U(1)'s and their effective couplings is performed both in field theory and string theory. It is motivated by the possible relevance of such couplings in particle physics, as well as a potential signal distinguishing string theory from other UV options. The most general anomaly related effective action is analyzed and parameterized. It contains Stückelberg, axionic and Chern-Simons-like couplings. It is shown that such couplings are generically non-trivial in orientifold string vacua and are *not* in general fixed by anomalies. A similar analysis in quantum field theories provides similar couplings. The trilinear gauge boson couplings are also calculated and their phenomenological relevance is advocated. We do not find qualitative differences between string and field theory in this sector.

Contents

1. Introduction and conclusions

It took some time for gauge and gravitational anomaly cancellation to take its place as a cornerstone in the building of theories of the fundamental interactions, [1]. Anomaly cancellation provides powerful constraints on chiral particle spectra. Gauge anomalies are intimately related to the UV structure of quantum field theories (QFTs). Their presence imply UV divergences that cannot be renormalized. There are several types of anomalies that plague gauge and gravitational theories. All of them are fatal in the UV of a QFT. However, their structure can be different in different cases. We therefore have (in 4d) pure non-abelian cubic anomalies, mixed anomalies between non-abelian and abelian gauge groups, as well as cubic abelian anomalies. In addition to this we have mixed abelian-gravitational anomalies associated with the trace of U(1) charges.

In string theory, the situation is slightly different. Closed string theory has a UV regime protected by the stringy cutoff introduced by the geometry of the closed Riemann surfaces. Modular invariance is crucial in this. It is the same invariance that guarantees the absence of irreducible (non-factorizable) anomalies. Reducible anomalies can be cancelled via the the Green-Schwarz mechanism [2], and its generalizations. Mixed abelian-non-abelian anomalies and cubic abelian anomalies are in this class. Generically, the chiral fermionic spectra in string models are not anomaly-free by themselves, but the gauge variation of their contribution to the one-loop effective action is precisely cancelled by antisymmetric tensor fields of various ranks which undergo non-linear gauge transformations [3]. In earlier perturbative heterotic constructions, the Green-Schwarz mechanism involves only one, universal, axion [4]. There have been recent discussions of heterotic compactifications in the supergravity limit, where the possibility of several anomalous U(1) factors was pointed out, [5]. This matches the situation in orientifold vacua [6, 7], which contain several axions or antisymmetric tensors [8, 9, 10, 11, 12]. A review can be found in [13].

In the presence of anomalous abelian factors in the gauge group, Stückelberg mixing with the axions render the "anomalous" gauge fields massive. The associated gauge symmetry is therefore broken. In the heterotic string, with a single anomalous U(1), such a mass is always fixed at the string scale [4]. The situation in orientifold vacua is richer and the masses depend non-trivially on volume and other moduli, allowing the physical masses of anomalous U(1) gauge bosons to be much smaller than the string scale, [14].

If the anomalous U(1) gauge boson masses are in the TeV range, they behave like Z' gauge bosons widely studied in the phenomenological literature [15]-[20]. One of the main points of this paper is that unlike other Z's discussed in heterotic string vacua as well as in unified models, the Z's associated to anomalous U(1)'s have other characteristic low-energy couplings. These are cubic couplings between various massive gauge bosons. Although their strength is of one-loop caliber, they can differentiate between different types of Z's.

An important role in our analysis is played by local gauge non-invariant terms in the effective action that we call generalized Chern-Simons terms (GCS), whose connection and role in the anomaly cancellation is one of the main goals of this paper. The presence of such couplings was first pointed out in [16], arising in the study of D-brane realizations of the Standard Model. They have been independently discovered in various supergravities in [21]-[24] and in higher dimensional gauge theories [25, 26].

In order to describe the relevant structure, we start from the anomaly-related terms in the effective action

$$\begin{aligned} \mathcal{S} = &-\sum_i \int d^4x \frac{1}{4g_i^2} F_{i,\mu\nu} F_i^{\mu\nu} - \frac{1}{2}\int d^4x \sum_I (\partial_\mu a^I + M_i^I A_\mu^i)^2 \ , \\ &+ \frac{1}{24\pi^2} C_{ij}^I \int a^I F^i \wedge F^j + \frac{1}{24\pi^2} E_{ij,k} \int A^i \wedge A^j \wedge F^k \ , \end{aligned} \tag{1.1}$$

where A_i are abelian gauge fields, a^I are axions with Stückelberg couplings which render massive (some of) the gauge fields and we used form language for compactness of notation in the last line of (1.1).

This action is gauge-variant under

$$A^i \to A^i + d\epsilon^i \quad , \quad a^I \to a^I - M_i^I \epsilon^i \tag{1.2}$$

This gauge-variance is tuned to cancel the anomalous variation of the one-loop effective action due to the standard triangle graphs. The contribution of the triangle graphs is scheme dependent, (see [27] and [28] as well as appendix A for a detailed exposition). In a natural scheme where the anomalous variation is distributed democratically among the three vertices, the anomaly cancellation conditions read

$$t_{ijk} \ + \ E_{ijk} + E_{ikj} \ + \ M_i^I \ C_{jk}^I \ = \ 0 \ . \tag{1.3}$$

Here $t_{ijk} = Tr(Q_i Q_j Q_k)$ are the standard anomaly traces and Q_i is the charge generator associated to A_i.

The GCS terms are known to be scheme dependent. However, the schemes that are relevant are typically model dependent, and it is more convenient to expose this asymmetry in the GCS terms explicitly. Moreover, there are combinations of GCS and axionic terms that are gauge invariant:

$$E_{IJk} \int (\partial a^I + M_i^I A^i) \wedge (\partial a^J + M_j^J A^j) \wedge F^k \tag{1.4}$$

Such gauge-invariant combinations lead to observable consequences.

An interesting related question is the following. String theory has been for a long time in search of a convincing, low energy signature of its existence. Despite several hints over the years it is fair to say that no such signature is known. The question can be posed as follows: considering the particle physics data up to a given energy (say LHC energies), is there a signature that would rule out a UV completion by an asymptotically free or asymptotically conformal QFT? Obviously we are keeping gravity out of this question as no QFT UV completion is known. Anomalous U(1)'s are ubiquitous in string theory, and it seems a good arena to search for such signatures.

The types of couplings we are investigating in this paper are related to triple gauge-boson couplings. For example, suppose there is a CP-odd three-boson coupling $Z' \, Z \, \gamma$. This may lead to a small but detectable experimental signal. Can a consistent renormalizable gauge theory lead to a similar effect? As we indicate, the answer turns out to be yes. We show this by generalizing the work of [29, 30]. In particular we consider the explicit example of a consistent chiral gauge theory that emerges after the decoupling of chiral fermions, charged with respect to the massive gauge fields and acquiring a large mass via Yukawa couplings to Higgs bosons.

The three-gauge-boson anomalous couplings we discuss in this paper have nontrivial consequences such as $Z' \to Z \, \gamma$ decays, which were not considered in the past in the context of Z' models [15, 18, 19]. A future detailed analysis of their experimental consequences would be important and could distinguish between models with standard anomaly cancellation and models with a generalized anomaly cancellation mechanism.

We summarize our results as follows:

- The starting point is a detailed analysis of a low-energy effective action (LEEA) which contains several U(1) gauge fields and axions. Some of the U(1) fields get a mass via Stückelberg couplings to some axions while others remain massless. The axions may be string theory RR axions or just the phases of Higgs field that break the gauge symmetry. We also include non-abelian gauge fields. There are two classes of gauge-non-invariant terms. The axionic couplings $C^I_{ij} a^I F^i \wedge F^j$ as well as the GCS terms $E_{ijk} A^i \wedge A^j \wedge F^k$. It is shown that the full anomaly related effective action is fully fixed by:

  (i) anomaly cancellation.

  (ii) The anomaly related change traces $t_{ijk} = Tr[Q_i Q_j Q_k]$

  (iii) The gauge invariant combinations of GCS and axionic terms in (1.4). From now on we will call these terms the "gauge-invariant GCS terms".

  The rest of the terms are determined by anomaly cancellation and depend on the scheme used to define the triangle contributions. We use a universal symmetric scheme, that has the advantage of being easy to use and model independent. It should be stressed that the gauge-invariant GCS terms are scheme-independent.

- We investigate in detail the structure of the anomaly-related effective action in orientifold models based on orbifold vacua of string theory, extending the analysis of [10]. In particular, we carefully compute, the disk coefficients M^I_i, ${C^I}_{ij}$ by a factorisation of one-loop data. Many details of the orbifold geometry are important in order to achieve this factorisation. In the process we explain the general procedure.

  We also compute the charge traces and verify that the GCS terms associated with antisymmetric pieces of $M^I_i {C^I}_{jk} - M^I_j {C^I}_{ik}$ are generically non-zero. We give the general algorithm for their computation, and provide detailed calculations in the Z_6 and Z_6' orientifolds. Moreover we show that the gauge-invariant GCS terms are generically non-zero, in string theory, as is the case in supergravity analyses [23]. This is a new result, as such terms are not fixed by anomalies.

- We compute the three-anomalous-gauge-boson one-loop open string amplitude and show explicitly that it is gauge invariant. This together with the disk couplings completes the string theory calculation of the relevant effective action.

- We analyze a similar situation in QFT. We consider a theory with an anomaly free set of chiral fermions, we give masses via Yukawa couplings to an anomalous subset of them, and compute the LEEFT at scales much smaller than the masses of heavy fermions. This LEEFT is of a similar kind to that of anomalous U(1)'s coming from string theory. We extend the previous computations of the anomalous effective action in [29, 30] to the general case, and derive the gauge-invariant GCS terms. They are generically non-trivial. The question of determining the UV charge spectrum from the low energy GCS terms does not have a unique solution. In particular an anomaly-free set of heavy fermions contributes non-trivially to the gauge-invariant GCS terms.

- We compute the full three-point amplitude at low energy of three U(1) gauge bosons. Some of them may be anomalous. Such amplitudes, although one-loop in strength, are important in characterizing the nature of Z' gauge bosons in colliders.

- We find no determining characteristic at low energy that would distinguish stringy anomalous U(1)'s from field theory effective anomalous U(1)s. This however is not exclusive. More analysis maybe necessary in this direction. An analogous question involves non-abelian symmetries. There are no known anomalous non-abelian symmetries in string theory. It is not clear whether there can be such effective symmetries in QFT. This question deserves further study.

The plan of our paper is as follows: In Section 2 we present a general analysis of the anomaly-related effective action with a generalized anomaly cancellation mechanisms, including the GCS terms. Section 3 explains the way the GCS couplings appear in orientifold models, gives a general criterion for their existence and formulae for axionic couplings and mixings.The explicit examples of the Z_6 and Z_6' orientifolds are analyzed in detail in appendix C. In section 4 we compute the relevant one-loop three gauge boson amplitude as well as the related by supersymmetry gauge-boson $\to$ 2 gaugini amplitude. In section five we compute the GCS and axionic terms emerging from integrating out massive fermions in QFT. Section 6 contains the calculation of triple effective gauge boson couplings.

In appendix A we review issues associated with the regularization, and calculation of triangle diagrams as well as their scheme dependence. Appendix B contains a collection of formulae relevant for the diagonalization of the arbitrary gauge boson action in described in section 2. Finally, Appendix D contains details of the computation of the full three gauge boson amplitude discussed in Section 6.

2. The general low-energy anomaly-related effective action

In this section we will perform a general analysis of the terms in the four-dimensional effective action relevant for anomaly cancellation. We will consider several anomalous U(1) vector bosons, A^i_μ, $i = 1, 2, \cdots, N_V$ with field strengths

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu \ . \tag{2.1}$$

We also consider non-abelian gauge bosons B_μ with non-abelian field strengths

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \ . \tag{2.2}$$

In orientifold vacua, both types of gauge fields will originate in the open sector. Finally, there will be a set of axion fields a^I, $I = 1, 2, \cdots, N_a$. Some will originate in the RR sector of the closed string sector while others will be the phases of open string charged scalars.

We will first start from the effective Lagrangian describing the kinetic terms of the fields

$$\begin{aligned}\mathcal{L}_{kin} = & -\frac{1}{2}\sum_{\alpha} f_{\alpha\beta}\mathrm{Tr}[G_{\alpha,\mu\nu}G_{\beta}^{\mu\nu}] - \frac{1}{4}\sum_{i,j} f_{ij}F_{i,\mu\nu}F_j^{\mu\nu} \\ & -\frac{1}{2}\sum_{I,J} h_{IJ}(\partial_\mu a^I + \sum_i M_i^I A_\mu^i)(\partial^\mu a^J + \sum_i M_i^I A^{i\mu})\end{aligned} \tag{2.3}$$

We have labeled the various simple factors of the non-abelian group[1] with the index $\alpha = 1, ...N_{YM}$, the Abelian factors with the index $i = 1, ...N_V$ and the axions with the index $I = 1, ...N_a$. From now on we will assume the summation convention: repeated indices are always summed over, unless otherwise stated.

In principle, the kinetic functions [2] $f_{\alpha\beta}$, f_{ij} and h_{IJ} depend on dilaton-like moduli φ. The dynamics of the latter is irrelevant for our present purposes and we can assume they are frozen at some non-singular value. At a given point in the moduli space, linear combinations of A_μ^i, A_μ^α and a^I put the kinetic terms in canonical form $f_{\alpha\beta} = \delta_{\alpha\beta}(1/g_\alpha^2)$, $f_{ij} = \delta_{ij}(1/g_i^2)$ and $h_{IJ} = \delta_{IJ}$. This results into a redefinition of the mixing coefficients $M_I^i \to \hat{M}_{\hat{I}}^{\hat{i}}$. Henceforth we assume we have performed this step and simply drop the hats.

Because of the mixing with the axions, a subset of the U(1) gauge bosons will eventually be massive. In string theory there are two sources for these Stückelberg couplings. The first is spontaneous symmetry breakdown ('Higgsing'), as in field theory. In this case M is proportional to the charge of the (Higgs) scalar obtaining a vev. The associated axion is the phase of the (open string) Higgs scalar. The second source of mixing ('axioning'), as in higher dimensional (supergravity) theories, emerges from the disk couplings between anomalous (open string) U(1) gauge fields and axions in the RR sector of the closed-string spectrum.

It is important for our subsequent purposes to separate the massive from the massless U(1) gauge fields. To implement this, we will diagonalize the mass matrix of the gauge bosons:

$$M_{ij}^2 \equiv M_i^I M_j^I \ . \tag{2.4}$$

In particular we will be careful to separate the zero eigenvalues. We will label by letters $m, n, \cdots$ the eigenvectors with zero eigenvalue, and with $a, b, \cdots$ the eigenvectors with non-zero eigenvalue.

$$M_{ij}^2 \ \eta_j^a = M_a^2 \ \eta_i^a \quad , \quad a = 1, 2, \cdots N_\bullet \quad , \quad M_a \neq 0 \ \ \forall \ a \ , \tag{2.5}$$

$$M_{ij}^2 \ \eta_j^m = 0 \quad , \quad m = 1, 2, \cdots N_{\rm o} \quad , \quad N_{\rm o} + N_\bullet = N_V \ . \tag{2.6}$$

The eigenvectors can be chosen to satisfy the orthonormality conditions

$$\eta_i^a \ \eta_i^b = \delta_{ab} \quad , \quad \eta_i^m \ \eta_i^n = \delta_{mn} \quad , \quad \eta_i^m \ \eta_i^a = 0 \quad , \quad M_i^I \ \eta_i^m = 0 \ . \tag{2.7}$$

[1] In our conventions $\mathrm{Tr}(t^A t^B) = 1/2\delta^{AB}$.

[2] Gauge invariance requires $f_{i\alpha} = 0$.

We also define the $N_{\rm o}$ vectors in the space of axions

$$W_a^I = \frac{M_i^I \eta_i^a}{M_a} \ . \tag{2.8}$$

This set is orthonormal using (2.7), (2.8)

$$W_a^I \ W_b^I = \frac{M_i^I \eta_i^a}{M_a} \frac{M_j^I \eta_j^b}{M_b} \frac{M_i^I M_j^I \eta_i^a \eta_j^b}{M_a M_b} = \delta_{ab} \ . \tag{2.9}$$

In general $N_a, N_V \geq N_\bullet$ so we may complete (2.9) into a full basis in axion space by introducing

$$W_u^I \ W_v^I = \delta_{uv} \quad , \quad u,v = 1,2\cdots, N_{\rm inv} = N_a - N_\bullet \quad , \quad W_u^I \ W_a^I = 0 \ . \tag{2.10}$$

We may use now the various vectors to define new fields as follows

$$A_i = \eta_i^a \ Q^a + \eta_i^m \ Y^m \quad , \quad a^I = W_u^I \ b^u + M_a W_a^I \ b^a \ . \tag{2.11}$$

The kinetic terms (2.3) in the new basis read

$$\begin{aligned}\mathcal{L}_{kin} = & -\frac{1}{2}{\rm Tr}[G_{\alpha,\mu\nu}G_\alpha^{\mu\nu}] - \frac{1}{4}F_{a,\mu\nu}F_a^{\mu\nu} - \frac{1}{4}F_{n,\mu\nu}F_n^{\mu\nu} \\ & -\frac{1}{2}\partial_\mu b^u \partial^\mu b^u - \frac{1}{2}M_a^2(\partial b^a + Q^a)^2 \ .\end{aligned} \tag{2.12}$$

Therefore, Q_μ^a denotes the $N_\bullet$ massive U(1) gauge fields, b^a their associated Stückelberg fields, Y_μ^m the $N_{\rm o}$ massless U(1) gauge fields, and b^u the $N_{\rm inv}$ gauge invariant axions.

The relevant infinitesimal U(1) gauge transformations are

$$Q_\mu^a \to Q_\mu^a + \partial_\mu \varepsilon^a \quad , \quad b^a \to b^a - \varepsilon^a \quad , \quad Y_\mu^m \to Y_\mu^m + \partial_\mu \varepsilon^m \ , \tag{2.13}$$

while the non-abelian ones read

$$B_\mu \to B_\mu + D_\mu \varepsilon \quad , \quad D_\mu \varepsilon \equiv \partial_\mu \varepsilon + [B_\mu, \varepsilon] \quad , \quad G_{\mu\nu} \to G_{\mu\nu} + [G_{\mu\nu}, \varepsilon] \ . \tag{2.14}$$

Under the above gauge transformations the kinetic terms are obviously invariant.

We will now introduce the classically gauge non-invariant terms of the effective action. Their ultimate goal will be to cancel the potential one-loop triangle anomalies. They are of two types. The first involves the Peccei-Quinn terms

$$\begin{aligned}\mathcal{L}_{PQ} = & \frac{b^u}{24\pi^2}({C^u}_{ab}F^a \wedge F^b + {C^u}_{am}F^a \wedge F^m + {C^u}_{mn}F^m \wedge F^n + {D^u}_\alpha Tr[G_\alpha \wedge G_\alpha]) \\ + & \frac{b^a}{24\pi^2}({C^a}_{bc}F^b \wedge F^c + {C^a}_{bm}F^b \wedge F^m + {C^a}_{mn}F^m \wedge F^n + {D^a}_\alpha Tr[G_\alpha \wedge G_\alpha]).\end{aligned} \tag{2.15}$$

$\mathcal{L}_{PQ}$ contains all possible Peccei-Quinn terms. We have used the form notation where

$$F = \frac{1}{2}F_{\mu\nu} \ dx^\mu dx^\nu \quad , \quad F \wedge F = \frac{1}{4}F_{\mu\nu}F_{\rho\sigma} \ dx^\mu dx^\nu dx^\rho dx^\sigma \ . \tag{2.16}$$

Under gauge transformations (2.13) and (2.14) the Peccei-Quinn terms transform as

$$\delta\mathcal{L}_{PQ} = -\frac{\epsilon_a}{24\pi^2}(C^a{}_{bc}F^b \wedge F^c + C^a{}_{bm}F^b \wedge F^m + C^a{}_{mn}F^m \wedge F^n + D^a{}_\alpha Tr[G_\alpha \wedge G_\alpha]) \quad (2.17)$$

The second set of gauge variant terms are the generalized Chern-Simons terms (or GCS terms for short). They are obtained by contracting the dual of the CS form with a gauge field. In the abelian case we may therefore write

$$\mathcal{S}^{ijk} \equiv \frac{1}{48\pi^2}\int \epsilon^{\mu\nu\rho\sigma} A^i_\mu A^j_\nu F^k_{\rho\sigma} \quad , \quad \mathcal{S}^{ijk} = -\mathcal{S}^{jik} \ . \quad (2.18)$$

Under U(1) gauge transformations, $A^i \to A^i + d\epsilon^i$

$$\delta\mathcal{S}^{ijk} = \frac{1}{24\pi^2}\int (\epsilon^j\ F^i \wedge F^k - \epsilon^i\ F^j \wedge F^k) \ . \quad (2.19)$$

Not all abelian GCS are independent. We have

$$\mathcal{S}^{ijk} + \mathcal{S}^{kij} + \mathcal{S}^{jki} = \frac{1}{48\pi^2}\int \epsilon^{\mu\nu\rho\sigma}\partial_\mu(A^i_\nu A^j_\rho A^k_\sigma) = 0 \ . \quad (2.20)$$

This relation indicates that when $i = k$ or $j = k$, there is a single independent GCS term. When all three indices are distinct, then there are two independent GCS terms.

To define the analogous GCS terms involving the non-abelian fields we introduce the standard non-abelian CS form

$$\Omega_{\mu\nu\rho} = \frac{1}{3}\mathrm{Tr}\left[B_\mu(G_{\nu\rho} - \frac{1}{3}[B_\nu, B_\rho]) + cyclic\right] \quad , \quad \frac{1}{2}\partial_\mu\Omega_{\nu\rho\sigma}\ dx^\mu dx^\nu dx^\rho dx^\sigma = \mathrm{Tr}[G \wedge G] \quad (2.21)$$

which under infinitesimal gauge transformations transforms as

$$\delta\Omega_{\mu\nu\rho} = \frac{1}{3}Tr\left[\partial_\mu\epsilon(\partial_\nu B_\rho - \partial_\rho B_\nu) + cyclic\right] \ . \quad (2.22)$$

Using the CS 3-form we may now construct the mixed GCS terms

$$\mathcal{S}^{i,\alpha} = \frac{1}{48\pi^2}\int \epsilon^{\mu\nu\rho\sigma} A^i_\mu \Omega^\alpha_{\nu\rho\sigma} \ . \quad (2.23)$$

Under infinitesimal abelian and non-abelian gauge transformations it transforms as

$$\delta\mathcal{S}^{i,\alpha} = \frac{1}{24\pi^2}\int F^i \wedge \mathrm{Tr}[\varepsilon\ \tilde{G}_\alpha] - \varepsilon^i \mathrm{Tr}[G_\alpha \wedge G_\alpha] \ , \quad (2.24)$$

where $\tilde{G}_\alpha$ is the abelian part of G_α. The most general set of GCS terms (irreducible under (2.20) is given by

$$\mathcal{L}_{GCS} = \frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\Big[E_{mnr}\ Y^m_\mu Y^n_\nu F^r_{\rho\sigma} + E_{man}\ Y^m_\mu Q^a_\nu F^n_{\rho\sigma} + E_{mab}\ Y^m_\mu Q^a_\nu F^b_{\rho\sigma}$$
$$+ E_{abc}\ Q^a_\mu Q^b_\nu F^c_{\rho\sigma} + (Z^m{}_\alpha\ Y^m_\mu + Z^a{}_\alpha\ Q^a_\mu)\Omega^\alpha_{\nu\rho\sigma}\Big] \ . \quad (2.25)$$

The coefficients satisfy the following symmetry properties

$$E_{mnr} = -E_{nmr} \quad , \quad E_{abc} = -E_{bac} \ . \quad (2.26)$$

The variation under infinitesimal gauge transformations takes the form

$$
\begin{aligned}
\int \delta\mathcal{L}_{GCS} = & \frac{1}{24\pi^2}\int E_{mnr}(\varepsilon^n F^m \wedge F^r - \varepsilon^m F^n \wedge F^r) + E_{man}(\varepsilon^a F^m \wedge F^n - \varepsilon^m F^a \wedge F^n) \\
& + E_{mab}(\varepsilon^a F^m \wedge F^b - \varepsilon^m F^a \wedge F^b) + E_{abc}\ (\varepsilon^b F^a \wedge F^c - \varepsilon^a F^b \wedge F^c) \\
& + Z^m{}_\alpha (F^m \wedge \mathrm{Tr}[\varepsilon \tilde{G}^\alpha] - \varepsilon^m \mathrm{Tr}[G^\alpha \wedge G^\alpha]) + Z^a{}_\alpha (F^a \wedge \mathrm{Tr}[\varepsilon \tilde{G}^\alpha] - \varepsilon^a \mathrm{Tr}[G^\alpha \wedge G^\alpha]) \\
= & \frac{1}{24\pi^2}\int \varepsilon^m \Big[-2E_{mnr}\ F^n \wedge F^r - E_{man}\ F^a \wedge F^n - E_{mab} F^a \wedge F^b - Z^m{}_\alpha \mathrm{Tr}[G^\alpha \wedge G^\alpha]\Big] \\
& + \varepsilon^a \Big[-2E_{abc}\ F^b \wedge F^c + E_{man}\ F^m \wedge F^n + E_{mab} F^m \wedge F^b - Z^a{}_\alpha \mathrm{Tr}[G^\alpha \wedge G^\alpha]\Big] \\
& + (Z^m{}_\alpha\ F^m + Z^a{}_\alpha F^a) \wedge \mathrm{Tr}[\varepsilon \tilde{G}^\alpha]\ .
\end{aligned} \tag{2.27}
$$

We may now consider the non-invariance of the effective action due to the anomalous triangle graphs. This is described in detail in appendix A. We use the totally symmetric scheme of defining the triangle graphs. We obtain the anomalous gauge variation

$$
\begin{aligned}
\int \delta\mathcal{L}_{\text{triangle}} = & -\frac{1}{24\pi^2}\int \Big[t_{abc}\varepsilon^a\ F^b \wedge F^c + t_{mnr}\varepsilon^m\ F^n \wedge F^r \\
& + t_{mab}(2\varepsilon^a\ F^b \wedge F^m + \varepsilon^m\ F^a \wedge F^b) + t_{amn}(2\varepsilon^m\ F^a \wedge F^n + \varepsilon^a\ F^m \wedge F^n) \\
& + T^a{}_\alpha (2\mathrm{Tr}[\varepsilon\ \tilde{G}^\alpha] \wedge F^a + \varepsilon^a\ \mathrm{Tr}[G^\alpha \wedge G^\alpha]) \\
& + T^m{}_\alpha (2\mathrm{Tr}[\varepsilon\ \tilde{G}^\alpha] \wedge F^m + \varepsilon^m\ \mathrm{Tr}[G^\alpha \wedge G^\alpha])\Big] \\
= & -\frac{1}{24\pi^2}\int \varepsilon^a \Big[t_{abc}\ F^b \wedge F^c + 2t_{mab}\ F^b \wedge F^m + t_{amn}\ F^m \wedge F^n + T^a{}_\alpha\ \mathrm{Tr}[G^\alpha \wedge G^\alpha]\Big] \\
& + \varepsilon^m \Big[t_{mnr}\ F^n \wedge F^r + t_{mab}\ F^a \wedge F^b + 2t_{amn}\ F^a \wedge F^n + T^m{}_\alpha\ \mathrm{Tr}[G^\alpha \wedge G^\alpha]\Big] \\
& + 2T^a{}_\alpha\ \mathrm{Tr}[\varepsilon\ \tilde{G}^\alpha] \wedge F^a + 2T^m{}_\alpha\ \mathrm{Tr}[\varepsilon\ \tilde{G}^\alpha] \wedge F^m\ .
\end{aligned} \tag{2.28}
$$

The tensors t and T are given by the cubic traces of the U(1) and non-abelian generators, $\mathcal{Q}_a$, $\mathcal{Q}_m$, T,

$$
\begin{aligned}
t_{abc} = \mathrm{Tr}[\mathcal{Q}_a\mathcal{Q}_b\mathcal{Q}_c] \quad , \quad t_{mab} = \mathrm{Tr}[\mathcal{Q}_a\mathcal{Q}_b\mathcal{Q}_m] \quad , \quad t_{amn} = \mathrm{Tr}[\mathcal{Q}_a\mathcal{Q}_m\mathcal{Q}_n]\ , \\
t_{mnr} = \mathrm{Tr}[\mathcal{Q}_m\mathcal{Q}_n\mathcal{Q}_r] \quad , \quad T^a{}_\alpha = \mathrm{Tr}[\mathcal{Q}_a(TT)^\alpha] \quad , \quad T^m{}_\alpha = \mathrm{Tr}[\mathcal{Q}_m(TT)^\alpha]\ ,
\end{aligned} \tag{2.29}
$$

with $(TT)^\alpha$ the quadratic Casimir of the α-th non-abelian factor. We have also assumed that the non-abelian cubic anomaly cancels.

Demanding gauge invariance of the total Lagrangian

$$
\mathcal{L} = \mathcal{L}_k + \mathcal{L}_{PQ} + \mathcal{L}_{GCS} + \mathcal{L}_{\text{triangle}}\ , \tag{2.30}
$$

we obtain the following conditions

$$
\begin{aligned}
E_{abc} + E_{acb} + C^a{}_{bc} + t_{abc} &= 0 \ , &(2.31)\\
-E_{mab} + C^a{}_{bm} + 2t_{mab} &= 0 \ , &(2.32)\\
-\frac{1}{2}(E_{man} + E_{nam}) + C^a{}_{mn} + t_{amn} &= 0 \ , &(2.33)\\
Z^a{}_\alpha + D^a{}_\alpha + T^a{}_\alpha &= 0 \ , &(2.34)\\
t_{mnr} + 2E_{mnr} &= 0 \ , &(2.35)\\
t_{mab} + \frac{1}{2}(E_{mab} + E_{mba}) &= 0 \ , &(2.36)\\
2t_{amn} + E_{man} &= 0 \ , &(2.37)\\
Z^m{}_\alpha + T^m{}_\alpha &= 0 \ , &(2.38)\\
2T^a{}_\alpha - Z^a{}_\alpha = 0 \quad , \quad 2T^m{}_\alpha - Z^m{}_\alpha &= 0 \ . &(2.39)
\end{aligned}
$$

Conditions (2.31)-(2.34) stem from the invariance under broken (massive) gauge transformations. Conditions (2.35)-(2.38) stem from the invariance under unbroken (massless) gauge transformations. Finally conditions (2.39) stem from nonabelian gauge invariance.

We now proceed to investigate some immediate implications of the invariance conditions above. (2.38) and (2.39) imply that

$$
Z^m{}_\alpha = T^m{}_\alpha = 0 \ , \tag{2.40}
$$

that is, the mixed abelian/non-abelian anomaly of the massless U(1)'s vanishes. This is indeed the case in all known orientifold examples. In (2.35) the tensor t_{mnr} is completely symmetric while E_{mnr} is antisymmetric in the first two indices. Therefore, this equation is only consistent if

$$
t_{mnr} = E_{mnr} = 0 \ . \tag{2.41}
$$

This implies that the massless U(1)'s should have no cubic anomaly among themselves. This is indeed the case in all known orientifold examples.

Solving (2.32), (2.33), (2.36), (2.37) we obtain

$$
E_{mab} + E_{mba} = -2t_{mab} \quad , \quad C^a{}_{bm} = -3t_{mab} + \frac{1}{2}(E_{mab} - E_{mba}) \tag{2.42}
$$

$$
, \quad E_{man} = \frac{2}{3} C^a{}_{mn} = -2t_{amn} \ . \tag{2.43}
$$

Solving (2.34) and (2.39) we obtain

$$
Z^a{}_\alpha = -\frac{2}{3} D^a{}_\alpha = 2T^a{}_\alpha \ . \tag{2.44}
$$

A counting of parameters in the anomaly equations is in order in order to motivate the general solution given below.

In equation (2.31), t has the symmetry $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}$ and therefore $\frac{N_\bullet(N_\bullet+1)(N_\bullet+2)}{3!}$ independent components. In appendix A we show that the tensor E has the symmetry $\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ and therefore

$\frac{N_\bullet(N_\bullet^2-1)}{3}$ independent components. C has the structure $\yng(2)\otimes\yng(1)$ and therefore $\frac{N_\bullet^2(N_\bullet+1)}{2}$ components. We have

$$\yng(2)\otimes\yng(1)=\yng(3)\oplus\yng(2,1) \tag{2.45}$$

Eqs (2.31) is a set of $\frac{N_\bullet^2(N_\bullet+1)}{2}$ independent equations.

In equations (2.32) and (2.36), t has $\frac{N_\circ N_\bullet(N_\bullet+1)}{2}$ independent components, while E and C have $N_\circ N_\bullet^2$ each. The number of independent equations is $N_\circ N_\bullet^2$ for (2.32) and $\frac{N_\circ N_\bullet(N_\bullet+1)}{2}^2$ for (2.36).

In equations (2.33) and (2.37), t and C have $\frac{N_\bullet N_\circ(N_\circ+1)}{2}$ independent components, while E has $N_\bullet N_\circ^2$. The number of independent equations is $\frac{N_\circ N_\bullet(N_\bullet+1)}{2}$ for (2.33) and $N_\circ N_\bullet^2$ for (2.37).

Finally, in equations (2.34) and the first of (2.39) all tensors have $N_\bullet N_n$ components, where N_n is the number of non-abelian group factors. This happens to also be the number of equations.

Equations (2.31)-(2.39) do not have a unique solution once the charges traces are fixed. The reason is the existence of the gauge invariant terms

$$\mathcal{L}_{inv}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(Q^a_\mu+\partial_\mu b^a)(Q^b_\nu+\partial_\nu b^b)\left[\mathcal{E}_{abc}F^c_{\rho\sigma}+\mathcal{E}_{mab}F^m_{\rho\sigma}\right] \ , \tag{2.46}$$

with $\mathcal{E}_{abc}=-\mathcal{E}_{bac}$, $\mathcal{E}_{mab}=-\mathcal{E}_{mba}$. $\mathcal{E}_{abc}$ has $\frac{N_\bullet(N_\bullet^2-1)}{3}$ independent components while $\mathcal{E}_{mbc}$, $\frac{N_\circ N_\bullet(N_\bullet-1)}{2}$.

By integrating by parts, we may reabsorb the various terms in (2.46) into $\mathcal{L}_{PQ}$ and $\mathcal{L}_{GCS}$. In particular, addition of $\mathcal{L}_{inv}$ to the effective action implies the following changes in $\mathcal{L}$

$$C^a{}_{bc}\to C^a{}_{bc}-\mathcal{E}_{abc}-\mathcal{E}_{acb} \quad , \quad E_{abc}\to E_{abc}+\mathcal{E}_{abc} \ , \tag{2.47}$$

$$C^a{}_{bm}\to C^a{}_{bm}-2\mathcal{E}_{mab} \quad , \quad E_{mab}\to E_{mab}-2\mathcal{E}_{mab} \ . \tag{2.48}$$

It is obvious from (2.31), (2.32) and (2.36) that such shifts leave the anomaly cancellation equations invariant. We should also remember that the PQ terms of the gauge-invariant axions are also gauge invariant. We may use this invariance to give the general solution to the anomaly cancellation equations (2.31)-(2.39).

Indeed, the general solution (E,C,Z,D) to the anomaly cancellation conditions can be written in terms of the charge trace tensors t_{abc}, t_{mab}, t_{amn}, $T^a{}_\alpha$, two arbitrary tensors $\mathcal{E}_{abc}$, $\mathcal{E}_{mab}$ satisfying $\mathcal{E}_{abc}=-\mathcal{E}_{bac}$, $\mathcal{E}_{mab}=-\mathcal{E}_{mba}$ as well as the PQ coefficients $C^M{}_{ab}$, $C^M{}_{am}$, $C^M{}_{mn}$ and $D^M{}_\alpha$. The general solution is

$$E_{abc}=\mathcal{E}_{abc} \quad , \quad E_{mab}=-t_{mab}+\mathcal{E}_{mab} \quad , \quad E_{man}=-2t_{amn} \ , \tag{2.49}$$

$$C^a{}_{bc}=-t_{abc}-\mathcal{E}_{abc}-\mathcal{E}_{acb} \quad , \quad C^a{}_{bm}=-3t_{mab}-\mathcal{E}_{mab} \quad , \quad C^a{}_{mn}=-3t_{amn} \tag{2.50}$$

$$Z^a{}_\alpha=2T^a{}_\alpha \quad , \quad D^a{}_\alpha=-3T^a{}_\alpha \ . \tag{2.51}$$

The counting of parameters that we presented above guarantees that this is the general solution.

The charge traces are computable from the classical action. Therefore, to fix the full low energy action, the $\mathcal{E}$ coefficients, undetermined from anomaly considerations must be calculated.

In orientifolds, this can be done by a disk calculation. To start with, the mixing coefficients M_i^I, which determine which U(1)'s become massive, are given by a disk two point function involving an open string vector and a closed string axion. Moreover, a disk three-point function, between two open-string vectors and a closed string axion determines the Peccei-Quinn C coefficients completely.[3] Once the C's have been determined, the unknown gauge invariant tensors can be evaluated as

$$\mathcal{E}_{abc} = \frac{1}{4}(C^b{}_{ac} - C^a{}_{bc}) \quad , \quad \mathcal{E}_{mab} = \frac{1}{4}(C^b{}_{am} - C^a{}_{bm}) \; . \tag{2.52}$$

We will do this in the next section for several orientifolds and show that the $\mathcal{E}$ tensors are generically non-zero. It should be noted that even in a theory free of four-dimensional anomalies (all cubic charge traces are zero) the gauge invariant GCS terms may be non-zero[4]. This is indeed the case in the theories of reference [23].

We will write here the general solution using the original arbitrary basis and the formulae of appendix B. In this basis the anomaly cancelling action is

$$\mathcal{L} = \mathcal{L}_{PQ} + \mathcal{L}_{GCS} \tag{2.53}$$

$$\mathcal{L}_{PQ} = \frac{C^I{}_{ij}}{24\pi^2}\; a^I \; F^i \wedge F^j + \frac{D^I{}_\alpha}{24\pi^2}\; a^I \; \mathrm{Tr}[G_\alpha \wedge G_\alpha] \tag{2.54}$$

$$\mathcal{L}_{GCS} = \frac{1}{48\pi^2}\epsilon^{\mu\nu\rho\sigma}\left[E_{ijk}\; A^i_\mu A^j_\nu F^k_{\rho\sigma} + Z^i{}_\alpha \; A^i_\mu \Omega^\alpha_{\nu\rho\sigma}\right] \; . \tag{2.55}$$

Using

$$Q^a = \eta^a_i A_i \quad , \quad Y^m = \eta^m_i A_i \quad , \quad b^u = W^I_u a^I \quad , \quad b^a = \frac{W^I_a}{M_a} a^I \; , \tag{2.56}$$

we find

$$\begin{aligned}
E_{ijk} &= \left[-(\tilde{G}^{ii'}G^{jj'} - G^{ii'}\tilde{G}^{jj'})\tilde{G}^{kk'} - \frac{1}{2}(\tilde{G}^{ii'}G^{jj'} - G^{ii'}\tilde{G}^{jj'})G^{kk'}\right] t_{i'j'k'} \\
&\quad + (\eta^a_i\eta^m_j - \eta^m_i\eta^a_j)\eta^b_k\mathcal{E}_{mab} + \eta^a_i\eta^b_j\eta^c_k\mathcal{E}_{abc} \; ,
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
C^I{}_{ij} &= W^I_u\left[C^u{}_{ab}\eta^a_i\eta^b_j + \frac{1}{2}C^u{}_{am}(\eta^a_i\eta^m_j + \eta^a_j\eta^m_i) + C^u{}_{mn}\eta^m_i\eta^n_j\right] \\
&\quad + \frac{W^I_a}{M_a}\left[C^a{}_{bc}\eta^b_i\eta^c_j + \frac{1}{2}C^a{}_{bm}(\eta^b_i\eta^m_j + \eta^b_j\eta^m_i) + C^a{}_{mn}\eta^m_i\eta^n_j\right] \\
&= -M^I_k\tilde{M}_{kk'}t_{i'j'k'}(G^{ii'}G^{jj'} + \frac{3}{2}(\tilde{G}^{ii'}G^{jj'} + G^{ii'}\tilde{G}^{jj'}) + 3\tilde{G}^{ii'}\tilde{G}^{jj'}) - 2\frac{M^I_k\eta^a_k\eta^b_i\eta^c_j}{M_a^2}\mathcal{E}_{abc} \\
&\quad - 2\frac{M^I_k\eta^a_k\eta^b_i\eta^m_j}{M_a^2}\mathcal{E}_{mab} + W^I_u\left[C^u{}_{bc}\eta^b_i\eta^c_j + C^u{}_{mb}\eta^m_i\eta^b_j + C^u{}_{mn}\eta^m_i\eta^n_j\right] \; ,
\end{aligned} \tag{2.58}$$

[3] There are subtleties with the normalization of the disk two and three-point functions, but they can be eventually resolved, eg by factorizing a non planar one-loop amplitude, to obtain an unambiguous answer.

[4] This may arise in a theory where a set of anomaly-free fermions has become massive due to the Higgs mechanism, and has been integrated out.

$$D^I{}_\alpha = D^u{}_\alpha W_u^I + D^a{}_\alpha \frac{M_i^I \eta_i^a}{M_a^2} \ , \tag{2.59}$$

$$Z^i{}_\alpha = Z^m{}_\alpha \eta_i^m + Z^a{}_\alpha \eta_i^a \ , \tag{2.60}$$

where $\tilde{M}_{kk'}$ was defined in (B.18).

It follows from (2.58) that

$$\begin{aligned} M_i^I C^I{}_{jk} = & -t_{i'j'k'} \left[G^{ii'} G^{jj'} G^{kk'} + \frac{3}{2}(G^{ii'} G^{jj'} \tilde{G}^{kk'} + G^{ii'} \tilde{G}^{jj'} G^{kk'}) + 3 G^{ii'} \tilde{G}^{jj'} \tilde{G}^{kk'} \right] \\ & -2 \eta_i^a \eta_j^b \eta_k^c \mathcal{E}_{abc} - \eta_i^a (\eta_j^b \eta_k^m + \eta_k^b \eta_j^m) \mathcal{E}_{mab} \ . \end{aligned} \tag{2.61}$$

Using $\tilde{G}^{ii'} \tilde{G}^{jj'} \tilde{G}^{kk'} t_{i'j'k'} = t_{mnr} \eta_i^m \eta_j^n \eta_k^r = 0$, we derive

$$t_{ijk} + E_{ijk} + E_{ikj} + M_i^I C_{jk}^I = 0 \ , \tag{2.62}$$

which is the condition for gauge invariance in a generic basis.

A remark concerns the GCS terms and their relation to the scheme dependence of the triangle anomalies. As we review in appendix A, all the scheme dependence of the triangle graphs is in one to one correspondence with the GCS terms. In particular, all the GCS terms can be set to zero in the effective action, by picking a particular scheme that treats the various U(1) factors asymmetrically. However in different orientifold ground states this scheme is vacuum dependent, as we show in the next section. We find it more convenient to fix once and for all, the fully symmetric scheme that treats all U(1)'s democratically and subsequently compute, and add to the effective action the GCS terms. This is what we do in the sequel.

We should stress, that the important effects that we discuss in this paper are gauge invariant and are therefore insensitive to the choice of scheme.

We should finally stress that the scheme dependence of the GCS terms can be also described by the non-uniqueness of the solution of the descent equations coming from the anomaly polynomial[5]. Restricting ourselves for simplicity to the abelian case, the anomaly polynomial is given by

$$I_6 \ = \ \frac{1}{6} t_{ijk} F^i F^j F^k \ = \ dI_5 \ , \tag{2.63}$$

where t_{ijk} is completely symmetric. The five-form I_5 is only defined up to a closed form. A solution is

$$I_5 \ = \ \frac{1}{6} t_{ijk} A^i F^j F^k \quad , \tag{2.64}$$

We may however add a closed form

$$\Delta I_5 = d(E_{ijk} A^i A^j F^k) \tag{2.65}$$

The gauge variation of I_5 defines the anomalies

$$\delta(I_5 + \Delta I_5) \ \equiv \ dI_4 \ = \ d \ (\frac{1}{6} t_{ijk} \epsilon^i F^j F^k \ + \ \frac{1}{3} E_{ijk} \epsilon^i F^j F^k) \ . \tag{2.66}$$

The scheme dependence is determined by the tensor E_{ijk} as advertised.

[5] M.B. and E.D. acknowledge B. Kors for a fruitful discussion on this and related issues in connection with the results of [31].

3. Anomalies and anomalous U(1)'s in orientifold models

It was shown in [9, 8] for 6d examples and in [10] for 4d orientifold vacua that the Green-Schwarz anomaly cancellation [2] in type II and orientifold vacua, generically involves twisted-form couplings to gauge fields. In [10] it was verified that indeed mixed abelian-non-abelian anomalies were cancelled by the twisted axions. Here we will discuss the subtleties that arise for the abelian sector.

In the previous section we have derived the anomaly cancellation condition (1.3),by utilizing a symmetric scheme for the triangle graphs. If

$$M^I_\alpha C^I_{\beta\gamma} - M^I_\beta C^I_{\alpha\gamma} \;\neq\; 0 \;, \tag{3.1}$$

then anomaly cancellation implies the existence of generalized Chern-Simons terms. For gauge groups coming from D-branes in type II orientifold models, this can arise only from a non-planar cylinder diagram that contains the (antisymmetrized) Chan-Paton traces will be (as we will see later in (D.21)):

$$3E_{\alpha\beta\gamma} = M^I_\alpha C^I_{\beta\gamma} - M^I_\beta C^I_{\alpha\gamma} = \sum_k \eta_k |\sqrt{N_k}| \; tr[\gamma_k \lambda_\gamma \lambda_{[\beta}] \; tr[\gamma_k \lambda_{\alpha]}] \;. \tag{3.2}$$

Here $k = 1 \cdots N-1$ denotes the different type of twisted sectors propagating in the tree-level channel cylinder diagram, whereas

$$N_k \;=\; \begin{cases} \prod_{\Lambda=1}^{3} (2\sin[\pi k v_\Lambda])^2 & \text{for } \mathrm{D9-D9} \text{ and } \mathrm{D5-D5} \text{ sectors,} \\ \\ (2\sin[\pi k v_3])^2 & \text{for } \mathrm{D9-D5} \text{ sectors} \end{cases} \tag{3.3}$$

denote the number of fixed points in the internal space and in the third internal torus, respectively (We consider for simplicity D5 branes whose world-volume span the third internal torus T^2_3.).

Also, η_k takes the values of: $\mathrm{sign}(\prod_{\Lambda=1}^{3} \sin[\pi k v_\Lambda])$ for all sectors of D9-D9, D5-D5, D9-D5 where the orbifold action twists all tori, $(-1)^{kv_i}$ for all sectors of D9-D5 where the orbifold action leaves untwisted a perpendicular torus T^2_i to the D5 brane (all the above are $\mathcal{N}=1$ sectors), and zero for sectors of D9-D9, D5-D5, D9-D5 where the orbifold action leaves untwists the longitudinal torus T^2_3 to the D5-brane (which are $\mathcal{N}=2$ sectors). Notice that particles and antiparticles contribute to the anomaly with different signs as it should be. Let us stress that the interpretation of the factors N_k is different for D9 and D5 branes. Whereas D9 branes fill the whole space-time and therefore couple to twisted axions localized at all fixed points, the D5 branes can only probe some fixed points and their associated axions. Correspondingly, their couplings to such axions are different with respect to the D9 brane couplings.

3.1 General formulae for the disk couplings of axions to gauge bosons

We would like to illustrate our results in some concrete examples such as type I compactifications on T^6/Z_N orbifolds. The resulting Chan-Paton group is typically non semi-simple and indeed contains one (for $N = 3,7$) or more abelian factors which are all superficially anomalous. When N is even, there are Z_2 elements $\mathcal{I}$ in the orbifold group. The $\Omega\mathcal{I}$ involution where Ω is the (generalized) world-sheet parity, generates $O5$-planes in the configuration. D5-branes are then needed for tadpole cancellation and the gauge group comprises two different kinds of gauge groups.

Denoting by γ the discrete Wilson lines, projectively embedding the orbifold group into the Chan-Paton group, we may parameterize them as follows:

$$\gamma_1^{(\alpha)} = \exp(-2\pi i \oplus_r V_r^{(\alpha)} \cdot H_r) \tag{3.4}$$

where H_r are the Cartan generators of $SO(32)^{(\alpha)}$ with $\alpha = 9,5$. They are normalized to $tr(H_r H_s) = 2\delta_{rs}$. For N odd, the conditions for (un)twisted RR tadpole cancellation are

$$tr[\gamma_{2k}^{(9)}] = 32 \prod_{\Lambda=1}^{3} \cos(\pi k v_\Lambda) \tag{3.5}$$

where i runs over the three two-tori. Both signs in $(\gamma_1^{(9)})^N = \pm 1$ are possible but the two choices lead to equivalent physics. For even N instead, only $(\gamma_1^{(9)})^N = (\gamma_1^{(5)})^N = -1$ is allowed. The form of the other twisted tadpole conditions is model dependent. Clearly $n_9 = n_5 = 32$ unless one turns on a quantized NS-NS antisymmetric tensor. Moreover $\Omega^2 = 1$, implies $(\gamma_\Omega^{(p)})^T = \pm\gamma_\Omega^{(p)}$ the standard choice is plus (+) for $p = 9$ and minus (−) for $p = 5$.

In order to study the fate of the anomalous U(1)'s, it is convenient to introduce the combinations

$$\lambda_i = \frac{1}{2\sqrt{n_i}} \sum_{r=1}^{n_i} Q_i^r H_r \tag{3.6}$$

where i denotes the brane and $Q_i^r = (0,0,\ldots,0,1,\ldots,1,0,\ldots,0)$ are 16-dimensional vectors, with n one-entries at the position where the corresponding $U(n)$ lives. Notice that λ's satisfy $tr[\lambda_i\lambda_j] = \frac{1}{2}\delta_{ij}$. Also

$$tr[\gamma_k\lambda_i] = -i\sqrt{n_i}\sin(2\pi k V_i) \quad tr[\gamma_k\lambda_i\lambda_j] = \frac{1}{2}\cos(2\pi k V_i)\delta_{ij} \ . \tag{3.7}$$

Notice that for $k = N/2$, the latter traces vanish since V_i are of the form $2\ell + 1/N$. This sector can only contribute with an internal volume dependent term associated to anomaly cancellation in $D = 6$.

The masses of the anomalous U(1)s have been computed in [14]. Here we review the results:

- $\mathcal{N} = 1$ Sectors: The contribution to the masses for $\mathcal{N} = 1$ sectors of Z_N orbifolds, labelled by k, are (we assume that the D5-branes are longitudinal to the T_3^2):

$$\frac{1}{2}M_{99,k}^2 = \frac{1}{2}M_{55,k}^2 = -\frac{1}{8\pi^3 N}\sqrt{N_k^1 N_k^2 N_k^3}\ tr[\gamma_k\lambda^a]tr[\gamma_k\lambda^b] \tag{3.8}$$

$$\frac{1}{2}M_{95,k}^2 = \frac{\tilde{\eta}_k}{8\pi^3 N}\sqrt{N_k^3}\ tr[\gamma_k\lambda^a]tr[\gamma_k\lambda^b] \tag{3.9}$$

where, $\tilde{\eta}_k$ is sign $\left(\prod_{\Lambda=1}^{3}\sin[\pi k v_\Lambda]\right)$ when all tori are twisted and (-1) when a perpendicular torus to the D5 brane remains untwisted by the orbifold action. Also $N_k^i = (2\sin[\pi k v_i])^2$ is the number of the effective fixed points of torus T_i^2.

- $\mathcal{N}=2$ Sectors: For such sectors, one $v_i k$ is integer *i.e.* one torus is untwisted by the orbifold action. This torus can be longitudinal or perpendicular to the D5 branes. Without loss of generality, we can assume that the longitudinal torus to the D5 brane is T_3^2 and the not untwisted-perpendicular one (if any) is T_2^2.

  Therefore, the contribution to the masses for $\mathcal{N}=2$, k sectors of Z_N orbifolds are:

$$\frac{1}{2}M^2_{99,k} \;=\; \frac{1}{2}M^2_{55,k,\parallel} \;=\; -\frac{2\mathcal{V}_3}{4\pi^3 N}\sqrt{N_k^1 N_k^2}\; tr[\gamma_k\lambda^a]tr[\gamma_k\lambda^b] \tag{3.10}$$

$$\frac{1}{2}M^2_{55,k,\perp} \;=\; -\frac{(2\mathcal{V}_2)^{-1}}{4\pi^3 N}\sqrt{N_k^1 N_k^3}\; tr[\gamma_k\lambda^a]tr[\gamma_k\lambda^b] \tag{3.11}$$

$$\frac{1}{2}M^2_{95,k,\parallel} \;=\; \frac{2\mathcal{V}_3}{4\pi^3 N}\;\tilde{\eta}_k\; tr[\gamma_k\lambda^a]tr[\gamma_k\lambda^b] \tag{3.12}$$

  where $\tilde{\eta}_k = (-1)^{kv_3}$ and $\mathcal{V}_i$ denotes the volume of the internal torus T_i^2. Notice that $\parallel$ and $\perp$ denote that the kth sector leaves invariant the longitudinal (third) or a perpendicular (second) torus to the D5 brane[6].

We extract the disc axionic couplings to a gauge boson M_a^I, for D9-branes by factorization of the one-loop mass matrix as follows

$$M^{k,f}_{a(9)}\Big|_{none} = \frac{i}{\sqrt{8\pi^3 N}}\sqrt{\ell_k^f}(N_k^1N_k^2N_k^3)^{-1/4}tr[\gamma_k\lambda_a] \qquad \forall f\in\mathcal{F}_k^{123} \tag{3.13}$$

$$M^{k,f}_{a(9)}\Big|_{T_3^2} = \frac{i\sqrt{2\mathcal{V}_3}}{\sqrt{4\pi^3 N}}\sqrt{\ell_k^f}(N_k^1N_k^2)^{-1/4}tr[\gamma_k\lambda_a] \qquad \forall f\in\mathcal{F}_k^{12} \tag{3.14}$$

where *none*, T_3^2 denotes the untwisted torus by the action of the kth sector of the orbifold. Notice also that we have split the sum over the index I labelling the various axions into a sum over sectors labelled by k and a sum over f, the 'effective' number of fixed points N_k. f spans the corresponding set $\mathcal{F}_k^{ij\ldots}$. Indices $ij...$ denote tori T_i^2, $T_j^2, ...$, where the fixed points are placed. D9 branes cover the entire space and pass through all fixed points. However, they couple differently to twisted axions which are living on these fixed points. This difference is denoted by ℓ_k^f, the length of the 'orbit' of fixed points which are identified under the orbifold action. In the case of geometric orientifolds of the type Z_N, ℓ_k^f takes the values:

$$\ell_k^f = \begin{cases} 1 & k \text{ sectors with } (k,N) \text{ coprime },\\ N/k & k \text{ sectors with } (k,N) \text{ non coprime and } k<[N/2] \end{cases} \tag{3.15}$$

where $[N/2]$ here is the integer part of $N/2$. In the last case in (3.15), we used the fact that sectors $N-k$ and k are equivalent and for $k<[N/2]$ and all supersymmetric compact

[6] As an example consider the Z_6' orientifold which has vector $v=(1,-3,2)/6$. Tadpole condition implies D9 branes and D5 branes which are longitudinal to the T_3^2. The $k=2,3$ are $\mathcal{N}=2$ sectors and the contribution to M_{55}^2 is given by (3.11), (3.10) respectively.

orbifolds, N/k, which counts the number of fixed points exchanged by orbifold operations, is integer for (k, N) non coprime.

For the case of D5-branes, the situation is even subtler because D5-branes couple to a reduced number of axions *i.e.* of fixed points. Here, we assume that D5-branes are longitudinal to the third torus T_3^2 and they are placed at the origin of the other two tori:

$$M_{a(5)}^{k,f}\Big|_{none} = \frac{i}{\sqrt{8\pi^3 N}} \left(\frac{N_k^1 N_k^2}{N_k^3}\right)^{1/4} tr[\gamma_k \lambda_a] \qquad \forall f \in \mathcal{F}_k^{123} \tag{3.16}$$

$$M_{a(5)}^{k,f}\Big|_{\perp} = \frac{i(1/\sqrt{2\mathcal{V}_2})}{\sqrt{4\pi^3 N}} \left(\frac{N_k^1}{N_k^3}\right)^{1/4} tr[\gamma_k \lambda_a] \qquad \forall f \in \mathcal{F}_k^{13} \tag{3.17}$$

$$M_{a(5)}^{k,f}\Big|_{\|} = \frac{i\sqrt{2\mathcal{V}_3}}{\sqrt{4\pi^3 N}} \left(N_k^1 N_k^2\right)^{1/4} tr[\gamma_k \lambda_a] \qquad \forall f \in \mathcal{F}_k^{12} \tag{3.18}$$

where *none*, $\|$ and $\perp$ denote that the kth sector leaves invariant none, the longitudinal or a perpendicular torus to the D5 brane respectively.

Similarly one can extract the disk axionic couplings to two bosons C_{ab}^I as follows

$$C_{ab(9)}^{k,f}\Big|_{none} = \frac{-i}{\sqrt{2\pi^3 N}} \sqrt{\ell_k^f} (N_k^1 N_k^2 N_k^3)^{-1/4} tr[\gamma_k \lambda_a \lambda_b] \qquad \forall f \in \mathcal{F}_k^{123} \tag{3.19}$$

$$C_{ab(9)}^{k,f}\Big|_{T_3^2} = \frac{-i\sqrt{2\mathcal{V}_3}}{\sqrt{4\pi^3 N}} \sqrt{\ell_k^f} (N_k^1 N_k^2)^{-1/4} tr[\gamma_k \lambda_a \lambda_b] \qquad \forall f \in \mathcal{F}_k^{12} \tag{3.20}$$

for D9 branes and

$$C_{ab(5)}^{k,f}\Big|_{none} = \frac{-i}{\sqrt{2\pi^3 N}} \left(\frac{N_k^1 N_k^2}{N_k^3}\right)^{1/4} tr[\gamma_k \lambda_a \lambda_b] \qquad \forall f \in \mathcal{F}_k^{123} \tag{3.21}$$

$$C_{ab(5)}^{k,f}\Big|_{\perp} = \frac{-i(1/\sqrt{2\mathcal{V}_2})}{\sqrt{4\pi^3 N}} \left(\frac{N_k^1}{N_k^3}\right)^{1/4} tr[\gamma_k \lambda_a \lambda_b] \qquad \forall f \in \mathcal{F}_k^{13} \tag{3.22}$$

$$C_{ab(5)}^{k,f}\Big|_{\|} = \frac{-i\sqrt{2\mathcal{V}_3}}{\sqrt{4\pi^3 N}} \left(N_k^1 N_k^2\right)^{1/4} tr[\gamma_k \lambda_a \lambda_b] \qquad \forall f \in \mathcal{F}_k^{12} \tag{3.23}$$

for D5 branes. The normalization of C^Is is such that all sectors contribute with the same footing in $M^I C^I$. By construction, in our Chan-Paton basis there are no mixed couplings between D5 and D9 brane anomalous U(1)'s.

4. String derivation of anomalous couplings

In this section we will sketch the string derivation of the anomalous three vector boson amplitude and argue that its anomalous variation cancels if RR tadpole cancellation takes place. Extracting the 'finite' CS terms turns out to be scheme dependent very much as in the effective field-theory, one should be able anyway to choose a 'symmetric' scheme. We will show how the axionic couplings can be unambiguously extracted and propose a natural prescription for identifying the relevant regions in the moduli space contributing to the triangle anomaly and to the GCS. We will also check that spacetime supersymmetry relates the GCS couplings to non-minimal couplings of two (neutral) ‘photinos’ to (anomalous) abelian vectors [23].

As shown by Green and Schwarz in their seminal paper [2] and confirmed by Inami, Kanno and Kubota [32] in a manifestly covariant approach, anomalous amplitudes in theories with open and unoriented string receive contribution from the boundary of the one-loop moduli space in the odd spin structure. This results from the subtle interplay between the presence of one 'supermodulus' (spin 3/2 'worldsheet gravitino' zero mode) and one conformal Killing spinor (spin $-1/2$ zero mode) [33]. The former brings down the worldsheet supercurrent $\mathcal{G} = \psi \cdot \partial X$ from the action or, equivalently, requires the insertion of $\delta(\beta) = e^{-\varphi}$ that absorbs the zero mode of the anti-superghost $\beta = e^{\varphi}\partial\xi$. The latter requires the insertion of $\delta(\gamma) = e^{+\varphi}$ that absorbs the zero mode of the superghost $\gamma = \eta e^{-\varphi}$ or, equivalently, allows to fix the position in superspace of one of the vertices, e.g. the 'longitudinal' one.

Focussing on potential gauge anomalies that only involve open string vectors and integrating over the single fermionic supermodulus ($\bar{\chi} = \pm\chi$, depending on the reflection / boundary conditions), one gets

$$\begin{aligned}\mathcal{A}_N(k_i, \zeta_i; \zeta_N = k_N) = &\int_0^\infty dt \int \prod_i dy_i \int d^2 z \chi_n^{(0)} \\ &\langle \mathcal{G}^n(z) \prod_i V(k_i, \zeta_i; y_i) V(k_N = -\sum_i k_i, \zeta_N = k_N; y_N)\rangle\end{aligned} \tag{4.1}$$

where $V(k_i, \zeta_i; y_i) = \zeta_i^\mu(\partial X_\mu + ik_i \cdot \psi\psi_\mu)e^{ik\cdot X}$ denote open string vertex operators (in the $q = 0$ superghost picture) $i = 1, ...N-1 = D/2$ (with even D). The last $i = N$ 'longitudinal' vertex operator $V(k_N, \zeta_N = k_N; y_N) = k_i^\mu \psi_\mu e^{ik\cdot X}$ can be expressed as a commutator $V(k_N, \zeta_N = k_N; y_N) = [\mathcal{Q}, e^{ik\cdot X}]$. Commuting the worldsheet supercharge $\mathcal{Q}$ through until one gets

$$[\mathcal{Q}, \mathcal{G}_n] = \gamma^m \mathcal{T}_{nm} \tag{4.2}$$

and relying on the conformal Ward identity for the insertion of the worldsheet stress tensor $\mathcal{T}_{nm}$ yields

$$\begin{aligned}\mathcal{A}_N(k_i, \zeta_i; \zeta_N k_N) = \mathcal{I}_{ab} N_a N_b &\int d^D X_0 d^D \psi_0 \int_0^\infty dt \times \\ &\frac{d}{dt}\prod_i dy_i \langle \prod_i V(k_i, \zeta_i; y_i) V(k_N = -\sum_{i\neq N} k_i, \zeta_N = k_N; y_N)\rangle \ ,\end{aligned} \tag{4.3}$$

where N_a and N_b are Chan-Paton multiplicities and $\mathcal{I}_{ab}$, the contribution of the sector (a, b) of the internal CFT, is a constant in the Ramond sector and coincides with the Witten index.

Integration over (non compact) bosonic and fermionic zero modes finally gives

$$\begin{aligned}\mathcal{A}_N(k_i, \zeta_i; \zeta_N = k_N) = &(2\pi)^D \delta^D(\sum_i k_i) \varepsilon_{\mu_1 ... \mu_{D/2}\nu_1 ... \nu_{D/2}} \zeta_1^{\mu_1} ... \zeta_{D/2}^{\mu_{D/2}} k_1^{\nu_1} ... k_{D/2}^{\nu_{D/2}} \mathcal{I}_{ab} N_a N_b \times \\ &\int_0^\infty dt \frac{d}{dt} \prod_i dy_i \langle e^{ik_i X} e^{ik_N X}\rangle\end{aligned} \tag{4.4}$$

Notice that most of the t dependence has cancelled between bosons and (periodic) fermions, which have the same (zero) modes thanks to the flatness of the surface.

Irreducible chiral anomalies are associated to amplitudes such that all vertex operators are inserted on the same boundary [34]. The planar contribution from the Annulus and the unorientable contribution from the Möbius strip cancel against one another after imposing RR tadpole cancellation in sectors with non-vanishing Witten index [12].

Reducible / factorizable anomalies are associated to non planar Annulus amplitudes such that insertions are distributed among the two boundaries. The divergence is regulated by momentum flow[7] and one can extract anomaly cancelling on shell couplings of closed string axions (p-forms) to open string 'composites'. This is the essence of the celebrated Green-Schwarz mechanism in $D = 10$ and its generalizations in lower dimensions [9, 34, 12]. Not without some effort, in $D = 4$, where $N = 3$, one can thus compute the PQ couplings C^I_{ij} and the mixing coefficients M^I_i, previously described. Whenever the combination $\sum_I C^I_{ij} M^I_k$ is not totally symmetric (*i.e.* generically) additional generalized Chern-Simons couplings (GCS) are required for the gauge invariance of the EFT description.

4.1 Direct computation

In principle, one could directly compute GCS in string theory in a similar way, e.g. by relaxing $\zeta_3 = k_3$. Thanks to the Killing supervector in the odd spin structure, one can still fix in superspace one of the insertions which amounts to using a vertex $\zeta_5 \cdot \psi e^{ik_3 X}$. Integration over ψ_0 then yields ($13 = 6 \times 5/2 - 2$) terms of different kinds depending on the choice of 4 out of 6 fermions, one from the supercurrent, one from the 'exotic' vertex and two each from the standard vertices, to soak up the 4 zero modes.

We have to evaluate:

$$\int_0^\infty \frac{dt}{t} \int_0^t dy_1 \int_0^{y_1} dy_2 \int d^2 z \ \times \tag{4.5}$$
$$\langle \zeta^1_\mu (\partial X^\mu_1 + ik_1 \psi_1 \psi^\mu_1) e^{ik_1 X_1} \ \zeta^2_\nu (\partial X^\nu_2 + ik_2 \psi_2 \psi^\nu_2) e^{ik_2 X_2} \ \zeta^3_\rho \psi^\rho_3 e^{ik_3 X_3} \ (\psi_{z,\lambda} \partial X^\lambda_z + \bar{\psi}_{z,\lambda} \bar{\partial} X^\lambda_z)\rangle$$

Since the internal CFT contributes 'topologically' to anomalous amplitudes, the relevant contractions involve only the non compact bosonic coordinates and their fermion partners in the odd spin structure. On the covering torus $\mathcal{T}$ the propagators are given by:

$$\begin{aligned} G(z,w) &= \langle X(z,\bar{z}) X(w,\bar{w})\rangle_{\mathcal{T}} = \frac{\alpha'}{2} \left(-\log \left| \frac{\vartheta_1(z-w|\tau)}{\vartheta'_1(0|\tau)} \right|^2 + 2\pi \frac{Im^2[z-w]}{Im[\tau]} \right) \\ S(z,w) &= \langle \psi(z,\bar{z}) \psi(w,\bar{w})\rangle_{\mathcal{T}} = -\partial_z G(z,w) \ . \end{aligned} \tag{4.6}$$

The latter is bi-periodic, but not analytic. For the Annulus with the involution $z \to \tilde{z} = 1 - \bar{z}$, one gets:

$$\langle X(z) X(w)\rangle_{\mathcal{A}} = \frac{1}{2}\Big(G(z,w) + G(z,\tilde{w}) + G(\tilde{z},w) + G(\tilde{z},\tilde{w}) \Big) = G(z,w) + G(z,\tilde{w}) \tag{4.7}$$

[7] When one of the two boundaries accomodates only one insertion, one needs to relax momentum conservation or to go slightly off-shell to regulate the amplitude.

since $G(z,w) = G(\tilde{z},\tilde{w})$.

We have to evaluate two kinds of terms. Terms with 4 worldsheet fermions and terms with 6 worldsheet fermions. The former (4 fermions terms) yield

$$\begin{aligned}
&\langle \zeta^1_\mu (\partial X^\mu_1) e^{ik_1 X_1}\ \zeta^2_\nu (ik_{2,\sigma}\psi^\sigma_2 \psi^\nu_2) e^{ik_2 X_2}\ \zeta^3_\rho \psi^\rho_3 e^{ik_3 X_3} \psi_{z,\lambda} \partial X^\lambda_z \rangle = \\
&\quad i\ \epsilon^{\sigma\nu\rho\lambda} \zeta^1_\mu\ p^2_\sigma\ \zeta^2_\nu\ \zeta^3_\rho\ \ e^{-k_1 k_2 G(1,2)}\ e^{-k_1 k_3 G(1,3)}\ e^{-k_2 k_3 G(2,3)} \times A \\
&\quad \times \Big[ik^\mu_2 \partial_1 G(1,2) + ik^\mu_3 \partial_1 G(1,3) \Big] \\
&\quad \times \Big(\Big[ik^\lambda_1 \partial_z G(z,1) + ik^\lambda_2 \partial_z G(z,2) + ik^\lambda_3 \partial_z G(z,3) \Big] + \eta^{\mu\lambda} \partial_z \partial_1 G(z,1) \Big)\ ,
\end{aligned} \tag{4.8}$$

where $A = t^{-2}$ comes from the normalization of the fermionic zero modes ($\sim t^{-1/2}$ each). A similar contribution is obtained with the exchange of 1 and 2.

The latter (6 fermion terms) yield

$$\begin{aligned}
&\langle \zeta^1_\mu (ik_{1,\kappa}\psi^\kappa_1 \psi^\mu_1) e^{ik_1 X_1}\ \zeta^2_\nu (ik_{2,\sigma}\psi^\sigma_2 \psi^\nu_2) e^{ik_2 X_2}\ \zeta^3_\rho \psi^\rho_3 e^{ik_3 X_3} \psi_{z,\lambda} \partial X^\lambda_z \rangle = \\
&\qquad ip^1_\kappa\ \zeta^1_\mu\ ip^2_\sigma\ \zeta^2_\nu\ \zeta^3_\rho\ \ e^{-k_1 k_2 G(1,2)}\ e^{-k_1 k_3 G(1,3)}\ e^{-k_2 k_3 G(2,3)} \times \\
&\qquad A \Big(S(1,2) [-\eta^{\kappa\sigma} \epsilon^{\mu\nu\rho\lambda} + \eta^{\kappa\nu} \epsilon^{\mu\sigma\rho\lambda} + \eta^{\mu\sigma} \epsilon^{\kappa\nu\rho\lambda} - \eta^{\mu\nu} \epsilon^{\kappa\sigma\rho\lambda}] \\
&\qquad\quad + S(1,3) [-\eta^{\kappa\rho} \epsilon^{\mu\sigma\nu\lambda} + \eta^{\mu\rho} \epsilon^{\kappa\sigma\nu\lambda}] + S(2,3) [-\eta^{\sigma\rho} \epsilon^{\kappa\mu\nu\lambda} + \eta^{\nu\rho} \epsilon^{\kappa\mu\sigma\lambda}] \\
&\qquad\quad + S(1,z) [\eta^{\kappa\lambda} \epsilon^{\mu\sigma\nu\rho} - \eta^{\mu\lambda} \epsilon^{\kappa\sigma\nu\rho}] + S(2,z) [\eta^{\sigma\lambda} \epsilon^{\kappa\mu\nu\rho} - \eta^{\nu\lambda} \epsilon^{\kappa\mu\sigma\rho}] \\
&\qquad\quad + S(3,z) \eta^{\rho\lambda} \epsilon^{\kappa\mu\sigma\nu} \Big) \Big(ik^1_\lambda \partial G(z,1) + ik^2_\lambda \partial G(z,2) + ik^3_\lambda \partial G(z,3) \Big)\ .
\end{aligned} \tag{4.9}$$

The explicit presence of two extra powers of momenta makes it clear that this contribution emerges from higher derivative string interactions.

In fact, the only potential low-derivative contribution correspond to the contraction of ∂X (in the worldsheet supercurrent) with $\partial X(i)$ in one of the two standard vertices. This yields

$$\begin{aligned}
\mathcal{A}(k_i,\zeta_i)\ =\ & (2\pi)^4 \delta^4 (\sum_i k_i) \varepsilon_{\mu\nu\rho\sigma} \zeta^\mu_1 \zeta^\nu_2 k^\rho_3 \zeta^\sigma_3 \times \\
& \int_0^\infty \frac{dt}{t^3} \int d^2 z \prod_i dy_i \partial_z \partial_1 G(z, y_1; t) \prod_{i<j} \exp(-k_i k_j G(i,j)) + (1 \leftrightarrow 2)\ .
\end{aligned} \tag{4.10}$$

The integration over z can be performed explicitly and yields a constant [35]. If momentum conservation were imposed, the subsequent integrations over y_i would simply yield powers of t since $k_i \cdot k_j = 0$ for three on-shell (massless) vectors. Relaxing momentum conservation, which is tantamount to postponing integration over the center of mass of the string X_0 until the very end, regulates the amplitudes and allows one to identify the various effective field theory contributions from the various 'corners' of the one-loop moduli space. Subtracting the residue of the simple pole at $t = 0$, that unambiguously yields the axionic exchange (closed string IR $\approx$ open string UV) and splitting the remaining t integral into two regions $I_{CS} = (0,T)$ and $I_{ta} = (T,\infty)$, it is easy to convince oneself that the latter exposes the triangle anomaly (open string IR) while the former exposes the GCS couplings that are generated by massive off-shell closed string exchange or, equivalently, by massive open strings circulating in the loop. As manifest in the need of introducing a cutoff T, the last two contributions cannot be separated unambiguously. Yet the total string amplitudes is clearly T independent. Any choice of T is a choice of scheme very much as in the effective field theory description.

4.2 The susy analog: $\gamma \to 2\tilde{\gamma}$

For supersymmetric theories, in the low energy effective description one also has [23] :

$$\mathcal{L}_{VFF} = E_{ij,k}\bar{\lambda}^j \sigma^\mu \lambda^k A^i_\mu + h.c. \tag{4.11}$$

with the same E's as in the GCS terms. This can be easily deduced in superspace, since both GCS and the VFF coupling arise from

$$E_{ij,k} \int d^4\theta V^i D^\alpha V^j W^k_\alpha + h.c. \tag{4.12}$$

Unfortunately this means that the corresponding one-loop VFF amplitude is also naively divergent / ambiguous. In fact it receives contribution both from the odd and the even spin structure that neatly combine to reproduce the GCS amplitude, up to obvious kinematic factors.

Our analysis shows that, at least for amplitudes with external fermions, the correct prescription is to insert one picture changing operator

$$\Gamma(z_0) = \{Q_{BRS}, \xi(z_0)\} = c\partial\xi + e^{+\varphi}G + \frac{1}{2}e^{+2\varphi}(b\partial\eta - 2\partial b\eta) \ , \tag{4.13}$$

rather than integrating over the supercurrent $\mathcal{G}$ insertion. The latter prescription would give unphysical branch cuts in this case while it gave an equivalent and thus correct result for the three vector amplitude above.

For this reason let us recall the form of the vertex operators in the relevant superghost pictures

$$\begin{aligned}
V^R_{-1/2} &= u^\alpha S_\alpha e^{-\varphi/2}\Sigma\ e^{ik\cdot X} \ , \\
V^R_{+1/2} &= \bar{v}^{\dot\alpha}\sigma^\mu_{\dot\alpha\alpha} S^\alpha \partial X_\mu e^{\varphi/2}\Sigma^+\ e^{ik\cdot X} + ... = \lim_{z_0\to z}\Gamma(z_0)V^R_{-1/2}(z) \ , \\
V^{NS}_0 &= \zeta^\mu(\partial X_\mu + ik\cdot\psi\psi_\mu)\ e^{ik\cdot X} \ , \\
V^{NS}_{-1} &= \zeta^\mu\psi_\mu e^{-\varphi}\ e^{ik\cdot X} \ .
\end{aligned} \tag{4.14}$$

The relevant contributions for amplitudes with a low number of insertions come from the action of the term proportional to $e^{+\varphi}\mathcal{G} = e^{+\varphi}(\psi^\lambda\partial X_\lambda + \mathcal{G}_{int})$ in $\Gamma(z_0)$. The internal spin fields that appear in the gaugino vertex operator can be bozonized as follows:

$$\Sigma = e^{i(\varphi_2+\varphi_3+\varphi_4)/2} \quad , \quad \Sigma^+ = e^{-i(\varphi_2+\varphi_3+\varphi_4)/2} \ . \tag{4.15}$$

In a given spin structure α, the amplitude that we are to evaluate is

$$A_{VFF} = \langle e^\varphi\psi\partial X(z_0)\ u^\alpha(k_1)S_\alpha e^{-\varphi/2}\Sigma e^{ik_1X_1}\ \bar{v}^{\dot\alpha}(k_2)C_{\dot\alpha}\Sigma^+e^{-\varphi/2}e^{ik_2X_2}\ \zeta_\mu(\partial X^\mu + ik_3\psi\psi^\mu)e^{ik_3X_3}\rangle_\alpha \tag{4.16}$$

Eventually one has to sum over both even and odd spin-structures with the GSO projection c_α.

The fermionic block is:

$$\langle e^{\varphi}\psi^\lambda e^{-\varphi/2}S_\alpha \Sigma e^{-\varphi/2}C_{\dot\alpha}\Sigma^+(\partial X^\mu + ik_3\psi\psi^\mu)\rangle \tag{4.17}$$

and the fermions can be bosonized as: $\psi^\lambda \to (+1,0)$, $S_\alpha \to (-1/2,-1/2)$, $C_{\dot\alpha} \to (-1/2,+1/2)$. Current algebra Ward identities also yield

$$\begin{aligned} ik^3_\nu\langle\psi^\lambda(0)S_\alpha(1) \quad C_{\dot\alpha}(2)\psi^\nu\psi^\mu(3)\rangle = ik^3_\nu\Big[&(\delta^\mu_\lambda\sigma^\nu_{\alpha\dot\alpha} - \delta^\nu_\lambda\sigma^\mu_{\alpha\dot\alpha})\partial_3 G(z_0-y_3)\\ &+\frac{1}{2}(\sigma^{\nu\mu})_\alpha^{\ \beta}(\sigma_\lambda)_{\beta\dot\alpha}\partial_3 G(y_3,y_1) + \frac{1}{2}(\sigma_\lambda)_{\alpha\dot\beta}(\sigma^{\nu\mu})^{\dot\beta}_{\ \dot\alpha}\partial_3 G(y_3,y_2)\Big] \end{aligned} \tag{4.18}$$

Finally the internal orbifold CFT contributes

$$\langle\Sigma(y_1)\Sigma^+(y_2)\rangle_\alpha \,=\, \vartheta_1(y_{12})^{-3/4}\vartheta_\alpha(\frac{y_{12}}{2})\prod_I\frac{\vartheta_\alpha(\frac{y_{12}}{2}+kv_I)}{\vartheta_1(kv_I)} \ , \tag{4.19}$$

where the effect of the orbifold projection (*viz.* kv_I) has been taken into account.

Assembling the various pieces above, up to the kinematical factor $u\sigma^\mu v\zeta_\mu$, one gets

$$\begin{aligned} &\frac{\vartheta_\alpha(y_{12}/2)}{\vartheta_1(y_{12})}\prod_{I=1}^3\frac{\vartheta_\alpha(y_{12}/2+kv_I)}{\vartheta_1(kv_I)} \times \Bigg[\Big\{\eta^{\lambda\nu}\partial_0\partial_3 G(z_0,y_3) + \sum_{i=1}^3 ik^\lambda_i\partial_z G(z_0,y_i)\ \sum_{j\neq 2}^3 ik^\nu_j\partial_2 G(2,j)\Big\}\\ &+\Big\{\frac{1}{2}(\sigma^\nu_\mu)^\beta{}_\alpha(\sigma_\lambda)_{\dot\beta\alpha}\left(\partial_3 G(y_3,y_1)-\partial_3 G(y_3,z_0)\right) - \frac{1}{2}(\sigma_\lambda)_{\alpha\dot\beta}(\bar\sigma^{\nu\mu})^{\dot\beta}{}_{\dot\alpha}\left(\partial_3 G(y_3,y_1)-\partial_3 G(y_3,z_0)\right)\Big\}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\sum_{i=1}^3 ik^\lambda_i\partial_0 G(z_0,y_i)\Bigg] \end{aligned} \tag{4.20}$$

Taking the limit $k_i \to 0$ and fixing the position of y_1, one has to perform the sum over the spin structures (at fixed twist structure)

$$\sum_{\alpha=1}^4 c_\alpha\int\frac{dt}{t^3}\int dy_2dy_3\frac{\vartheta_\alpha(y_{12}/2)}{\vartheta_\alpha(y_{12})}\prod_{I=1}^3\frac{\vartheta_\alpha(y_{12}/2+kv_I)}{\vartheta_\alpha(kv_I)}\partial_0\partial_3 G(z_0,y_3)\ . \tag{4.21}$$

For c_α the coefficients of the GSO-projection, one can make use of the identity:

$$\frac{1}{2}\sum_{\alpha\beta=0}^1(-1)^{\alpha+\beta+\alpha\beta}\prod_{i=1}^4\vartheta[^\alpha_\beta](v_i) = -\prod_{i=1}^4\vartheta_1(v'_i)\ ,$$

where $v'_i = -v_i + \frac{1}{2}\sum_l v_l$, and find that

$$\frac{1}{2}\sum_\alpha\vartheta_\alpha(y_{12}/2)\prod_I\vartheta_\alpha(y_{12}/2+kv_I) = -\vartheta(y_{12})\prod_I\vartheta_\alpha(kv_I)\ ,$$

and therefore all ϑ_αs exactly cancel. One ends up with an amplitude similar to the one for the insertion of three bosonic VO's, that in the same $k_i \to 0$ limit reads

$$\sim \int \frac{dt}{t^3} \int dy_2 dy_3 \partial_0 \partial_3 G(z_0, y_3) \ . \tag{4.22}$$

Once again extracting the susy counterpart of the GCS is scheme dependent but one can unambiguously identify (supersymmetrized) axion exchange with the residue of the simple pole at $t = 0$. Introducing an open string IR cutoff T as above, one can associate the contribution of the susy partners of the GCS to the interval $t = (0, T)$ and the 'massless' open string loop with the region $t = (T, \infty)$. One has to keep in mind that only the total sum is T independent and thus unambiguous, the individual contributions are non gauge invariant and thus ambiguous (scheme dependent).

5. Heavy fermions and low-energy effective actions

So far we have analyzed in detail the structure of the anomaly-related effective action for orientifold models. We have seen, that apart from the generic appearance of anomalous U(1)'s, there is a rich pattern of axionic couplings and GCS terms. It is an interesting question if such patterns emerge in EFTs of UV-complete[8] quantum field theories. In particular, we are interested in knowing, whether in the anomaly sector of an EFT, we can distinguish whether the UV completion is stringy or a UV-complete QFT.

To proceed we consider a consistent (*i.e.* anomaly-free) and renormalizable gauge theory with spontaneously-broken gauge symmetry via the Brout-Englert-Higgs mechanism[9]. Through appropriate Yukawa couplings, some large masses to a subset of the fermions can be given. We denote by $\psi^{(H)}_{L,R}$ such massive chiral fermions. Their $U(1)_i$ charges are $X^{(H)i}_{L,R}$. In the sequel, we will generalize the [29, 30] calculations of the effective anomaly related couplings in the EFT, generated by the loops of the heavy chiral fermions.

The relevant terms in the effective action of the heavy fermion sector of the theory are

$$\begin{aligned} L_H \ &= \ \bar{\psi}^{(H)}_L \left(i\gamma^\mu \partial_\mu + X^{(H)i}_L \gamma^\mu A^i_\mu \right) \psi^{(H)}_L + \bar{\psi}^{(H)}_R \left(i\gamma^\mu \partial_\mu + X^{(H)i}_R \gamma^\mu A^i_\mu \right) \psi^{(H)}_R \\ &\quad - \left(\lambda^H_I \phi_I \bar{\psi}^{(H)}_L \psi^{(H)}_R + \text{h.c.} \right) \ , \end{aligned} \tag{5.1}$$

where ϕ^I are a set of Higgs fields of $U(1)_i$ charges X^i_I. They spontaneously break the abelian gauge symmetries via their vevs, $\langle \phi^I \rangle$ We are interested in a chiral fermion set,

$$X^{(H)i}_L \ - \ X^{(H)i}_R \ = X^i_I \neq \ 0 \ . \tag{5.2}$$

If the associated Yukawa couplings are large, $\lambda^H_I \gg g_i$, spontaneous symmetry breaking generates large Dirac fermion masses $M_H = \lambda^H_I v_I$, where $\langle \phi_I \rangle = v_I$. We consider the heavy

[8]We define a QFT to be UV-complete if all gauge couplings are asymptotically free or asymptotically conformal.

[9]Strictly speaking quartic scalar couplings necessary for the Higgs potential are IR free. We will still call this a UV complete theory as the scalars could be bound states of fermions.

fermion decoupling limit, with fixed Higgs vev's and fixed gauge boson masses, whereas $M_H \to \infty$. If the initial theory were anomaly-free, i.e.

$$\sum_l (X_L^i X_L^j X_L^k - X_R^i X_R^j X_R^k)^{(l)} + \sum_H (X_L^i X_L^j X_L^k - X_R^i X_R^j X_R^k)^{(H)} \;=\; 0 \;, \qquad (5.3)$$

where (l) denote the massless (light) fermionic spectrum, then in the low-energy theory (*i.e.* at energies below the heavy fermion masses) with the heavy fermions integrated-out, there are Adler-Bell-Jackiw triangle anomalies coming from the light fermions. There will also be Wess-Zumino-like couplings generated by the loops of the heavy fermions. The resulting low-energy action, for simple gauge groups, was worked out in specific regularization schemes in various papers starting with [30]. After symmetry breaking, we parameterize the scalar fields by

$$\phi_I \;=\; (v_I + h_I)\; e^{\frac{i\; a_I}{v_I}} \;, \qquad (5.4)$$

where h_I are massive Higgs-like fields and a_I are gauge-variant phases (axions) which will play a crucial role in the anomaly cancellation at low-energy. To be definite, we consider a larger number of abelian gauge fields than gauge-variant axions. The gauge transformations of gauge fields and axions are

$$\delta A_\mu^i \;=\; \partial_\mu \epsilon^i \quad , \quad \delta a_I \;=\; v_I\; X_I^i\; \epsilon^i \;, \qquad (5.5)$$

where X_I^i are the $U(1)_i$ charges of ϕ_I.

We can compute explicitly the eventual GCS terms by performing a diagrammatic computation starting from the action (5.1). In order to do this, we start from the corresponding three gauge boson amplitude induced by triangle diagram loops of heavy fermions and expand in powers of external momenta k_i/M_H. We work in a basis of left (L) and right (R) fermionic fields, with the fermionic propagator having the components

$$S_{LL}(p) = S_{RR}(p) = \frac{-\not{p}}{p^2 - M_H^2} \quad , \quad S_{LR}(p) = \frac{-M_H}{p^2 - M_H^2} \;. \qquad (5.6)$$

The purely left and right propagation in the triangle loop is similar to the computation of the anomaly with massive fermions in the loop and will only be sketched here. The corresponding contribution to the three-point function is of the form [10]

$$\begin{aligned}\Gamma_{ijk}^{\nu\rho\mu}(p,k_1,k_2,a) \;&=\; i \sum_H (X_R^i X_R^j X_R^k)\; tr\Big[\; \frac{\not{p} - \not{k}_1 + \not{a}}{(p-k_1+a)^2 - M_H^2}\gamma^\nu \frac{\not{p} + \not{a}}{(p+a)^2 - M_H^2} \\ &\times \gamma^\rho\; \frac{\not{p} + \not{k}_2 + \not{a}}{(p+k_2+a)^2 - M_H^2}\gamma^\mu \frac{1+\gamma_5}{2}\;\Big] \;, \end{aligned}\qquad (5.7)$$

for the right-handed fermions, where a is the shift vector [28], and a similar expression with obvious changes for left-handed ones. Only the linear terms in the expansion do correspond to GCS terms. By expanding to linear order we obtain

$$\Gamma_{ijk}^{\nu\rho\mu}(p,k_i,a) \;\simeq\; \Gamma_{ijk}^{\nu\rho\mu}(p,0,0) \;+\; k_i^\alpha \frac{\partial}{\partial k_i^\alpha}\Gamma_{ijk}^{\nu\rho\mu}(p,k_i,0)|_{k_i 0} \;+\; a^\alpha \frac{\partial}{\partial a^\alpha}\Gamma_{ijk}^{\nu\rho\mu}(p,0,a)|_{a=0} \;, \qquad (5.8)$$

[10]See Appendix A for notations and conventions for triangle diagrams.

the shift vector is parameterized as

$$a^{\alpha}_{ijk} = A_{ijk} k_1^{\alpha} + B_{ijk} k_2^{\alpha} \ , \tag{5.9}$$

A straightforward computation indicates that the term in the effective action, originating from the second term in the r.h.s. of (5.8) is proportional to

$$t^{(H)}_{ijk,L-R} \int (A^i \wedge A^j \wedge F^k + A^i \wedge A^k \wedge F^j) \tag{5.10}$$

where

$$t^{(H)}_{ijk,L-R} = (X^i_L X^j_L X^k_L)^{(H)} - (X^i_R X^j_R X^k_R)^{(H)} \ . \tag{5.11}$$

It vanishes identically, since $t^{(H)}_{ijk,L-R}$ is symmetric in all indices, whereas the GCS terms are antisymmetric in two indices. On the other hand, the last term in the r.h.s. of (5.8) gives a surface contribution in the loop momentum p. The surface integral is evaluated to be

$$\int d^4p \ \partial_\sigma \frac{p^2 p_\epsilon}{(p^2 - M_H^2)^3} \ = \ -\frac{\pi^2}{4} \eta_{\sigma\epsilon} \ . \tag{5.12}$$

The contribution to the effective action therefore is

$$S^{(1)}_{GCS} = \frac{1}{48\pi^2} \ \sum_H t^{(H)}_{ijk,L-R} \ \int (A_{ijk} A^i \wedge A^k \wedge F^j \ - \ B_{ijk} A^i \wedge A^j \wedge F^k) \ . \tag{5.13}$$

We observe from (5.13) that the contribution to the GCS terms coming from diagrams without mass insertions are zero in the natural scheme in which the anomaly is split democratically between the different external currents (the symmetric scheme).

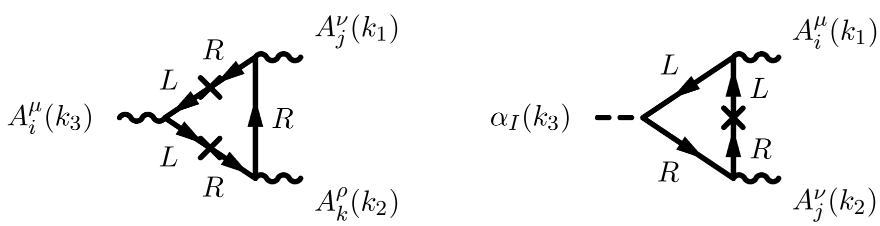


Figure 1: The first diagram is one of the twelve diagrams which contribute to the GCS terms. The second is one of the six diagrams which contribute to the axionic couplings. Both are obtained by integrating out heavy fermions.

The new interesting ingredients in the massive fermion case appear due to the mass insertions in the propagators $S_{LR}(p)$. Mass insertions on two of the three fermionic propagators produce new contributions which are UV finite and easily evaluated. There are twelve new diagrams corresponding to the three possible ways of distributing the mass insertions, to the symmetrization of the external bosonic lines and to the two types of components (left versus right-handed fermions) in each propagator. We portray just one example. For mass insertions on the propagators of momenta $p - k_1$ and $p + k_2$, one of the contributions to the three-point function is

$$\Gamma^{\nu\rho\mu}_{ijk}(p,k_1,k_2)^{(2)} = i\sum_H (X_L^i X_R^j X_R^k)^{(H)}\ tr\ \Big[\frac{M_H\gamma^\nu}{(p-k_1)^2-M_H^2}\frac{\not p\gamma^\rho}{p^2-M_H^2}\frac{M_H\gamma^\mu}{(p+k_2)^2-M_H^2} + \frac{M_H}{(p-k_2)^2-M_H^2}\gamma^\rho\frac{\not p}{p^2-M_H^2}\gamma^\nu\frac{M_H}{(p+k_1)^2-M_H^2}\gamma^\mu\ \frac{1+\gamma_5}{2}\ \Big]\ . \quad (5.14)$$

Since the result is finite, we don't need to introduce a shift vector a. As before, we expand in powers of the external momenta and we keep only the linear term. By a straightforward computation we find

$$\int \frac{d^4p}{(2\pi)^4}\Gamma^{\nu\rho\mu}_{ijk}(p,k_1,k_2)^{(2)} = \epsilon^{\mu\nu\rho\alpha}(k_2-k_1)_\alpha \sum_H \frac{4M_H^2}{3}(X_L^i X_R^j X_R^k)^{(H)}\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2-M_H^2)^3}\ . \quad (5.15)$$

The result (5.15) gives a finite contribution in the $M_H\to\infty$ limit. By adding the twelve different diagrams, we find a local term in the effective action

$$S^{(2)}_{GCS} = \frac{1}{96\pi^2}\sum_H (X_L^i X_R^j - X_R^i X_L^j)^{(H)}(X_R^k + X_L^k)^{(H)}\int A_i\wedge A_j\wedge F_k\ . \quad (5.16)$$

The lagrangian (5.1) contains also couplings of axions to the fermions, of the type

$$L_{\text{Yuk}}\ =\ -i\ \lambda_I^H a_I \bar\psi^{(H)}\gamma_5\psi^{(H)} + \cdots\ , \quad (5.17)$$

where the ellipsis stands for higher order couplings that give no contributions in the $M_H\to\infty$ limit. The axion-heavy fermion couplings generate axionic couplings to gauge fields through one mass insertion in triangle diagrams. There are six relevant diagrams. We consider as an example the one with the mass insertion on the fermionic propagator of momentum $p+k_2$. One of the two contributions to the three-point function $\tilde\Gamma^{\mu\nu}_{ij}(p,k_1,k_2)^{(1)}$ is equal to [11]

$$i\sum_{H_I}\lambda_I^{H_I}(X_L^i X_L^j)^{(H_I)}\ tr\ \Big[\ \gamma_5\frac{\not p-\not k_1}{(p-k_1)^2-M_{H_I}^2}\gamma^\mu\frac{\not p}{p^2-M_{H_I}^2}\frac{M_{H_I}}{(p+k_2)^2-M_{H_I}^2}\gamma^\nu\frac{1-\gamma_5}{2}\ \Big]\ . \quad (5.18)$$

It leads to

$$\int\frac{d^4p}{(2\pi)^4}\tilde\Gamma^{\mu\nu}_{ij}(p,k_1,k_2)^{(1)}\epsilon^{\mu\nu\alpha\beta}(k_1)_\alpha(k_2)_\beta\sum_{H_I}\lambda_I^{H_I}M_{H_I}(X_L^i X_L^j)^{(H_I)}\int\frac{d^4p}{(2\pi)^4}\frac{p^2}{(p^2-M_{H_I}^2)^4}\ . \quad (5.19)$$

We take the limit $\lambda_I^{H_I}\to\infty$ with fixed v_I. In this limit (5.19) survives and is proportional to $1/v_I$. Adding the five other diagrams we get the axionic couplings

$$S_{\text{ax}} = \frac{1}{96\pi^2}\sum_I\sum_{H_I}[2(X_L^i X_L^j + X_R^i X_R^j) + X_L^i X_R^j + X_R^i X_L^j]^{(H_I)}\ \int\ \frac{a_I}{v_I}\ F_i\wedge F_j\ . \quad (5.20)$$

[11]The heavy fermions H obtain their mass generically from a single Higgs, whose phase is a_I. We denote this by H_I. Therefore, a sum over H_I is over all massive fermions who get their mass from the I-th Higgs.

On the other hand, the kinetic terms of the Higgs fields ϕ_I generate the Stückelberg mixings

$$|\partial_\mu \phi_I - i X_I^i A_\mu^i \phi_I|^2 \to (\partial_\mu a_I - X_I^i v_I A_\mu^i)^2 \ . \tag{5.21}$$

We therefore find the following GCS terms, axionic couplings and kinetic mixings

$$\begin{aligned} E_{ij,k} &= \frac{1}{4} \sum_H (X_L^i X_R^j - X_R^i X_L^j)^{(H)} (X_R^k + X_L^k)^{(H)} \ , \\ C_{ij}^I &= \frac{1}{4 v_I} \sum_{H_I} [2(X_L^i X_L^j + X_R^i X_R^j) + X_L^i X_R^j + X_R^i X_L^j]^{(H_I)} \ , \\ M_i^I &= v_I \ X_I^i \ = \ v_I \ (X_L^i - X_R^i)^{(H_I)} \quad , \quad \text{for every } H_I \ . \end{aligned} \tag{5.22}$$

There are some obvious checks of the formulae above. Clearly the GCS terms should cancel in the non-chiral case $X_L^i = X_R^i$. In the particular chiral case $X_L^i = -X_R^i$, they should also cancel since the GCS terms have to be antisymmetric under the left-right interchange $X_L^i \leftrightarrow X_R^i$ and the first term in the GCS terms in (5.22) is already antisymmetric. Notice the following identities which prove the anomaly cancellation conditions are satisfied as a particular case of the more general analysis performed in Section 2

$$\begin{aligned} \frac{1}{3}(M_i^I C_{jk}^I + M_j^I C_{ki}^I + M_k^I C_{ij}^I) &= \frac{1}{2} \sum_H (X_L^i X_L^j X_L^k - X_R^i X_R^j X_R^k)^{(H)} \ , \qquad (5.23) \\ \frac{1}{3}(M_i^I C_{jk}^I - M_j^I C_{ik}^I) &= \frac{1}{4} \sum_H (X_L^i X_R^j - X_R^i X_L^j)^{(H)} (X_R^k + X_L^k)^{(H)} \ = \ E_{ij,k} \ . \end{aligned}$$

Moreover, in the case where all the U(1)'s are massive, comparison with (2.49-2.51) indicates that the gauge invariant GCS are given by the second line above.

The gauge variations of the induced GCS terms and axionic couplings in (5.16) and (5.20) are

$$\delta(S_{GCS} + S_{\rm ax}) \ = \frac{1}{48\pi^2} \ \sum_H (X_L^i X_L^j X_L^k - X_R^i X_R^j X_R^k)^{(H)} \int \epsilon_i F_j \wedge F_k \ . \tag{5.24}$$

This anomalous variation is 1/2 compared to the standard anomaly contribution in the appendix, (A.14). The reason is that (5.24) is not yet the full anomaly (see e.g. [25]). Indeed, the classical value of the divergence of the heavy fermionic current is

$$\begin{aligned} \partial^\mu J_\mu^{(H)} &= i M_H (X_R^i - X_L^i)^{(H)} \ \bar{\psi}^{(H)} \gamma_5 \psi^{(H)} \ , \\ \text{where } J_\mu^{(H)} &= X_R^{i,(H)} \bar{\psi}_R^{(H)} \gamma_\mu \psi_R^{(H)} + X_L^{i,(H)} \bar{\psi}_L^{(H)} \gamma_\mu \psi_L^{(H)} \ . \end{aligned} \tag{5.25}$$

The matrix element of this classical part can be evaluated diagramatically. The computation is basically identical to the computation of the axionic couplings above. The result, in the decoupling limit $M_H \to \infty$, is

$$\begin{aligned} &\langle 0 \mid i \ M_H \ (X_R^i - X_L^i)^{(H)} \ \bar{\psi}^{(H)} \gamma_5 \psi^{(H)} | A_j^\mu(k_1) A_k^\nu(k_2) \ \rangle \ = \qquad (5.26) \\ &\frac{1}{48\pi^2} (X_R^i - X_L^i)^{(H)} [2(X_L^j X_L^k + X_R^j X_R^k) + X_L^j X_R^k + X_R^j X_L^k]^{(H)} \epsilon^{\mu\nu\alpha\beta} (k_1)_\alpha (k_2)_\beta \ . \end{aligned}$$

When subtracted in order to define the real anomaly, the result in (5.24) is multiplied by a factor of two and can correctly be cancelled by the anomalies of the massless (light) fermionic spectrum, by using the initial anomaly cancellation conditions (5.3). Although $iM_H(X_R^i - X_L^i)^{(H)}\ \bar{\psi}^{(H)}\gamma_5\psi^{(H)}$ is not an operator in the effective theory after integrating out the heavy fermions, its effects can be accounted for by doubling the axionic couplings (5.22). In doing so, the anomalies are cancelled up to local GCS terms. As we already discussed, the coefficient of the GCS terms can always be changed by scheme redefinition. The simplest scheme is the one in which the anomaly is democratically distributed among the anomalous currents in the light fermionic loops. In this scheme, the GCS terms and axionic couplings are given by

$$\begin{aligned}
E_{ij,k} &= \frac{1}{2}\sum_H (X_L^i X_R^j - X_R^i X_L^j)^{(H)}(X_R^k + X_L^k)^{(H)} \ , \\
C_{ij}^I &= \frac{1}{2v_I}\ \sum_{H_I}[2(X_L^i X_L^j + X_R^i X_R^j) + X_L^i X_R^j + X_R^i X_L^j]^{(H_I)} \ .
\end{aligned} \tag{5.27}$$

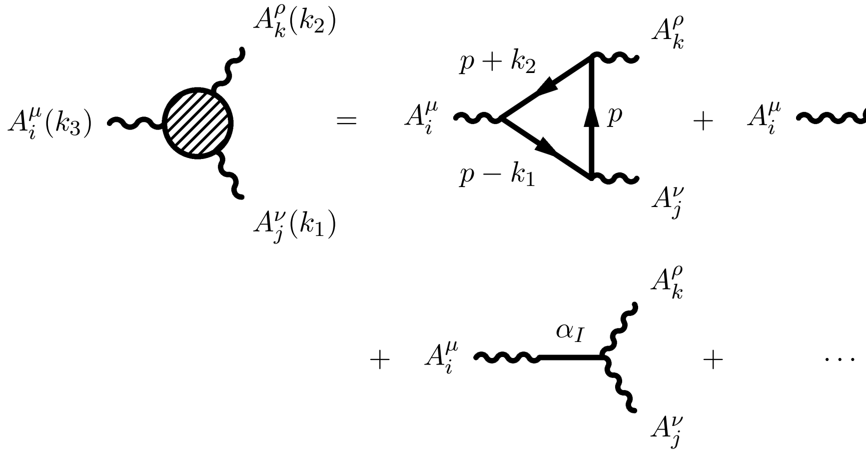
It is important to emphasize that, while the GCS terms are scheme dependent due to (5.13), the axionic couplings (5.27) are UV finite and therefore scheme independent.

Up to now, we have discussed in this section only massive gauge fields, which look superficially anomalous at low energy due to our ignorance about the high-energy anomalous set of heavy fermions $\psi^{(H)}$. The formalism easily incorporates massless and non-anomalous gauge bosons A_m, which are defined by the necessary (but not sufficient) condition that the Higgs fields ϕ_I be neutral $X_I^m = 0$, which implies in our renormalizable examples that the heavy fermions are non-chiral $X_L^m = X_R^m$. As a result one has $E_{mn,p} = 0$ in agreement with our previous findings.

In conclusion, the decoupling of heavy chiral fermions by large Yukawa couplings does generate a generalized Green-Schwarz mechanism at low energy, with axionic couplings canceling anomalies of the light fermionic spectrum. It also leads to generalized Chern-Simons terms which play an important role in anomaly cancellation, in analogy to the string orientifold models we analyzed in the previous sections.

A very important and interesting question is the comparison of the results of this section with the string theory results of the previous sections and try to find possible differences. If possible, this would be a remarkable way to distinguish between low-energy predictions of string theory versus 4d field theory models. We were not able however to find such a difference. Our present conclusion is therefore that any experimental signature like anomalous three-boson couplings at low energy is a strong hint towards, either an underlying string theory with generalised anomaly cancellation mechanism, or a standard renormalizable field theory with very heavy chiral fermions, which generate a similar anomaly cancellation pattern.

6. Three gauge boson amplitudes



(6.1)

There are three diagrams (6.1) to evaluate : the anomalous triangle diagrams, the tree-level axionic exchange ones and the ones coming from the contact GCS terms. In the following we define: $t^{ijk} = \sum_f [Q_f^i Q_f^j Q_f^k]$. The triangle amplitude in (6.1), in momentum space, is given by

$$\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = i^3 t^{ijk} \int \frac{d^4p}{(2\pi)^4} \frac{Tr[\gamma_\mu(\not{p}+\not{k}_2)\gamma_\rho \not{p}\gamma_\nu(\not{p}-\not{k}_1)\gamma_5]}{(p+k_2)^2(p-k_1)^2p^2} \tag{6.2}$$

and can be decomposed according to

$$\begin{aligned}\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = \; & t^{ijk}[A_1(k_1,k_2)\epsilon_{\mu\nu\rho\sigma}k_2^\sigma + A_2(k_1,k_2)\epsilon_{\mu\nu\rho\sigma}k_1^\sigma + B_1(k_1,k_2)k_{2\nu}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau \\ & + B_2(k_1,k_2)k_{1\nu}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau + B_3(k_1,k_2)k_{2\rho}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau + B_4(k_1,k_2)k_{1\rho}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau]\end{aligned} \tag{6.3}$$

where A's and B's functions of k_1, k_2.

The coefficients A_i, B_i are computed in Appendix D. The three different contributions are given by

$$\Gamma^{ijk}_{\mu\nu\rho} = \Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} + \Gamma^{ijk}_{\mu\nu\rho}|_{axion} + \Gamma^{ijk}_{\mu\nu\rho}|_{CS} \ , \tag{6.4}$$

where

$$\begin{aligned}\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = \; & t^{ijk}[A_1\epsilon_{\mu\nu\rho\sigma}k_2^\sigma + A_2\epsilon_{\mu\nu\rho\sigma}k_1^\sigma + B_1k_{2\nu}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau + B_2k_{1\nu}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau \\ & + B_3k_{2\rho}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau + B_4k_{1\rho}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau] \ ,\end{aligned} \tag{6.5}$$

$$\begin{aligned}\Gamma^{ijk}_{\mu\nu\rho}|_{axion} = \; & -M_I^iC_I^{jk}\left(\frac{k_{3\mu}}{k_3^2}\right)\epsilon_{\nu\rho\sigma\tau}k_2^\sigma k_1^\tau - M_I^jC_I^{ki}\left(\frac{k_{1\nu}}{k_1^2}\right)\epsilon_{\rho\mu\tau\sigma}k_2^\sigma k_3^\tau - M_I^kC_I^{ij}\left(\frac{k_{2\rho}}{k_2^2}\right)\epsilon_{\mu\nu\tau\sigma}k_3^\sigma k_1^\tau \\ = \; & -M_I^iC_I^{jk}\frac{-(k_{1\mu}+k_{2\mu})}{(k_1+k_2)^2}\epsilon_{\nu\rho\sigma\tau}k_2^\sigma k_1^\tau + M_I^jC_I^{ki}\frac{k_{1\nu}}{k_1^2}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau - M_I^kC_I^{ij}\frac{k_{2\rho}}{k_2^2}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau\end{aligned} \tag{6.6}$$

$$\begin{aligned}\Gamma^{ijk}_{\mu\nu\rho}|_{CS} = \; & -E^{ij,k}\epsilon_{\mu\nu\rho\sigma}k_2^\sigma - E^{jk,i}\epsilon_{\nu\rho\mu\sigma}k_3^\sigma - E^{ki,j}\epsilon_{\rho\mu\nu\sigma}k_1^\sigma \\ = \; & -(E^{ij,k}-E^{jk,i})\epsilon_{\mu\nu\rho\sigma}k_2^\sigma - (E^{ki,j}-E^{jk,i})\epsilon_{\mu\nu\rho\sigma}k_1^\sigma \ .\end{aligned} \tag{6.7}$$

As shown in Appendix D, by using the anomaly cancellation conditions (D.20), we can eliminate the scheme-dependent coefficients A_i in terms of the finite and unambiguous coefficients B_i. The final result is

$$\begin{aligned}
\Gamma^{ijk}_{\mu\nu\rho} \;=\;& \left[-t_{ijk}(\frac{C_A}{3}+k_1k_2B_1+k_1^2B_2)-E^{ij,k}+E^{jk,i}\right]\epsilon_{\mu\nu\rho\sigma}k_2^\sigma \\
&+\left[t_{ijk}(\frac{C_A}{3}+k_1k_2B_1-k_2^2B_3)-E^{ki,j}+E^{jk,i}\right]\ \epsilon_{\mu\nu\rho\sigma}\ k_1^\sigma \\
&+\left[t_{ijk}(\frac{C_A}{3}\frac{k_{1\nu}}{k_1^2}+k_{1\nu}B_2+k_{2\nu}B_1)+E^{ij,k}-E^{jk,i}\frac{k_{1\nu}}{k_1^2}\right]\ \epsilon_{\mu\nu\sigma\tau}\ k_2^\sigma k_1^\tau \\
&+\left[t_{ijk}(-\frac{C_A}{3}\frac{k_{2\rho}}{k_2^2}-k_{1\rho}B_1+k_{2\rho}B_3)+E^{ki,j}-E^{jk,i}\frac{k_{2\rho}}{k_2^2}\right]\ \epsilon_{\mu\nu\sigma\tau}\ k_2^\sigma k_1^\tau\ . \qquad (6.8)
\end{aligned}$$

The upshot of our analysis is that in the case of a generalized anomaly cancellation mechanism, there are anomalous three-gauge boson couplings at low energy (6.8). These couplings involve at least one massive, anomalous gauge fields, which we call generically Z' in what follows. These new couplings could be tested at LHC if the masses of the anomalous , Z' gauge bosons, are small enough, in the TeV range. This is possible in orientifold models, but not only [25], especially in the case of a low fundamental string scale. The best signature of these anomalous couplings are the $Z' \to Z\ \gamma$ decays, which to our knowledge were never considered in phenomenological Z' models. A more detailed analysis is clearly needed in to study the experimental consequences of these decays in future collider experiments and particularly at LHC.

There is an interesting way to analyze the effect of GCS terms in the CP-odd part of the three gauge-boson amplitude, in terms of its analytic structure. Following Coleman and Grossman [36], for the simple kinematical configuration

$$k_1^2\ = k_2^2\ =\ k_3^2\ \equiv\ Q^2\ , \qquad (6.9)$$

the one-loop triangle contribution to the 3-boson amplitude is

$$\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = -\frac{1}{3Q^2}\ t^{ijk}C_A\ [\ \epsilon_{\nu\rho\sigma\tau}(k_1^\mu+k_2^\mu)+\epsilon_{\mu\rho\sigma\tau}k_1^\nu-\epsilon_{\mu\nu\sigma\tau}k_2^\rho\ ]\ k_2^\sigma k_1^\tau\ . \qquad (6.10)$$

By adding the triangle diagram, the axionic exchange which are both *non-local* and the *local* GCS contributions and after using the gauge invariance conditions, we find the total result

$$\begin{aligned}
\Gamma^{ijk}_{\mu\nu\rho}|_{total}\ =\ & \Gamma^{ijk}_{\mu\nu\rho}|_{CP=even} \\
&+\frac{1}{Q^2}\,[\epsilon^{\nu\rho\sigma\tau}(E_{ki,j}-E_{ij,k})(k_1^\mu+k_2^\mu)+\epsilon^{\mu\rho\sigma\tau}(E_{ij,k}-E_{jk,i})k_1^\nu \\
&+\epsilon^{\mu\nu\sigma\tau}(E_{ki,j}-E_{jk,i})k_2^\rho]\,k_2^\sigma k_1^\tau \\
&-E_{ij,k}\epsilon^{\mu\nu\rho\sigma}k_2^\sigma+E_{jk,i}\epsilon^{\nu\rho\mu\sigma}(k_1^\sigma+k_2^\sigma)-E_{ki,j}\epsilon^{\rho\mu\nu\sigma}k_1^\sigma\ . \qquad (6.11)
\end{aligned}$$

Notice that the pole in $1/Q^2$ is completely determined by the GCS terms and does not exist if they are absent.

Acknowledgments

It is a pleasure to thank Luis Alvarez-Gaume, Sergio Ferrara, Boris Kors, Costas Kounnas, Augusto Sagnotti, Timo Weigand and Arkady Vainshtein for useful discussions. M. B. and E. D. would like to thank the Galileo Galilei Institute for Theoretical Physics for hospitality and INFN for partial support during completion of this work. This work was supported in part by the CNRS PICS no. 2530 and 3059, INTAS grant 03-516346, MIUR-COFIN 2003-023852, NATO PST.CLG.978785, the RTN grants MRTN-CT-2004-503369, EU MRTN-CT-2004-512194, MRTN-CT-2004-005104 and by a European Union Excellence Grant, MEXT-CT-2003-509661. P. A. was supported by the research program "Pythagoras II" (grant 70-03-7992) of the Greek Ministry of National Education, partially funded by the European Union.

APPENDIX

A. Triangle anomalies and regularization dependence

In this appendix we will present some known facts about triangle graphs and scheme dependence. They are useful in our general analysis of the effective action. We use the conventions of Weinberg's textbook, [28] to which we refer the reader for all details that we omit here.

We use a basis for the fermions so that they are all left-handed. We will package them into a single spinor ψ. The associated charge operator for the gauge field A^i_μ is denoted by $\mathcal{Q}_i$. We define the various U(1) currents as

$$J^\mu_i = -i\bar{\psi}\mathcal{Q}_i\gamma^\mu\psi \tag{A.1}$$

The three-current correlator we will study is

$$\Gamma^{\mu\nu\rho}_{ijk}(x,y,z) = \langle J^\mu_i(x) J^\nu_i(y) J^\rho_i(z)\rangle \tag{A.2}$$

The leading contribution, at one loop emerges from fermions going around the loop. The total contribution is obtained by summing over all relevant fermion fields.

There are two diagrams for the correlator that can be evaluated to yield

$$\begin{aligned}\Gamma^{\mu\nu\rho}_{ijk}(x,y,z) =& -i\mathrm{Tr}\left[S(x-y)Q_j\gamma^\nu P_L S(y-z)Q_k\gamma^\rho P_L S(z-x)Q_i\gamma^\mu P_L\right]\\ & -i\mathrm{Tr}\left[S(x-z)Q_k\gamma^\rho P_L S(z-y)Q_j\gamma^\nu P_L S(y-x)Q_i\gamma^\mu P_L\right]\end{aligned} \tag{A.3}$$

with

$$P_L = \frac{1+\gamma^5}{2} \quad , \quad S(x) = -\int\frac{d^4p}{(2\pi)^4}\frac{\not{p}}{p^2}e^{ip\cdot x} \tag{A.4}$$

Substituting we obtain

$$\begin{aligned}\Gamma^{\mu\nu\rho}_{ijk}(x,y,z) =& \; it_{ijk}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}e^{-i(k_1+k_2)\cdot x+ik_1\cdot y+ik_2\cdot z}\int\frac{d^4p}{(2\pi)^4}\times\\ & \left\{\mathrm{Tr}\left[\frac{\not{p}-\not{k}_1+\not{a}}{(p-k_1+a)^2}\gamma^\nu\frac{\not{p}+\not{a}}{(p+a)^2}\gamma^\rho\frac{\not{p}+\not{k}_2+\not{a}}{(p+k_2+a)^2}\gamma^\mu P_L\right]+\right.\\ & \left.+\mathrm{Tr}\left[\frac{\not{p}-\not{k}_2+\not{b}}{(p-k_2+b)^2}\gamma^\rho\frac{\not{p}+\not{b}}{(p+b)^2}\gamma^\nu\frac{\not{p}+\not{k}_1+\not{b}}{(p+k_1+b)^2}\gamma^\mu P_L\right]\right\}\end{aligned} \tag{A.5}$$

with $t_{ijk} = \mathrm{Tr}[\mathcal{Q}_i\mathcal{Q}_j\mathcal{Q}_k]$ We have shifted the integrated momentum in the two diagrams using two vectors a_μ and b_μ. This reflects the standard ambiguity of the triangle graph and translates into the definition of the associated current opeartors. Demanding that there is no anomaly in the vector currents forces $b=-a$, choice that we keep from now on. The vector a is parameterizing the leftover scheme dependence of the triangle graph in question.

We may now obtain the following divergence formulae,

$$\partial_\mu\Gamma^{\mu\nu\rho}_{ijk}(x,y,z) = -\frac{t_{ijk}}{8\pi^2}\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}e^{-i(k_1+k_2)\cdot x+ik_1\cdot y+ik_2\cdot z}\;\epsilon^{\nu\rho\sigma\tau}\;a_\sigma(k_1+k_2)_\tau \tag{A.6}$$

$$\partial_\nu \Gamma^{\mu\nu\rho}_{ijk}(x,y,z) = -\frac{t_{ijk}}{8\pi^2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} e^{-i(k_1+k_2)\cdot x + ik_1\cdot y + ik_2\cdot z}\ \epsilon^{\mu\rho\sigma\tau}\ (a+k_2)_\sigma (k_1)_\tau \quad \text{(A.7)}$$

$$\partial_\rho \Gamma^{\mu\nu\rho}_{ijk}(x,y,z) = -\frac{t_{ijk}}{8\pi^2}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4} e^{-i(k_1+k_2)\cdot x + ik_1\cdot y + ik_2\cdot z}\ \epsilon^{\mu\nu\sigma\tau}\ (k_1-a)_\sigma (k_2)_\tau \quad \text{(A.8)}$$

A generic choice of scheme (i.e. a_μ) indicates that the divergence structure is asymmetric among the three vertices of the triangle graph. There is a single choice that is fully symmetric, namely

$$a = \frac{1}{3}(k_1 - k_2) \quad \text{(A.9)}$$

We now proceed to construct the effective action for the gauge fields after integrating out the fermions. To cubic order we obtain

$$S_{ijk} = \frac{1}{3!}\int d^4x\ d^4y\ d^4z\ \Gamma^{\mu\nu\rho}_{ijk}(x,y,z)\ A^i_\mu(x)\ A^j_\nu(y)\ A^k_\rho(z) \quad \text{(A.10)}$$

where no summation is assumed on the i,j,k labels.

Upon gauge transformations $A^i_\mu \to A^i_\mu + \partial_\mu \varepsilon^i$ we obtain

$$\delta S_{ijk} = -\frac{1}{3!}\int \ \left[\varepsilon^i \partial_\mu \Gamma^{\mu\nu\rho}_{ijk} A^j_\nu(y)\ A^k_\rho(z) + \varepsilon^j \partial_\nu \Gamma^{\mu\nu\rho}_{ijk} A^i_\mu(x)\ A^k_\rho(z) + \varepsilon^k \partial_\rho \Gamma^{\mu\nu\rho}_{ijk} A^i_\mu(x)\ A^j_\nu(y)\right] \quad \text{(A.11)}$$

If a is a constant independent of momenta then it does not contribute to the gauge variations. We therefore parameterize the scheme dependence as

$$a = Ak_1 + Bk_2 \quad \text{(A.12)}$$

The real numbers A, B can be different for different ijk combinations.

$$\begin{aligned}\delta S_{ijk} \ &= \ -\frac{t_{ijk}}{3!(32\pi^2)}\int d^4x\ \Big\{(A-B)_{ijk}\varepsilon^i\ \epsilon^{\mu\nu\rho\sigma}\ F^j_{\mu\nu}F^k_{\rho\sigma} + (B_{ijk}+1)\varepsilon^j\ \epsilon^{\mu\nu\rho\sigma}\ F^i_{\mu\nu}F^k_{\rho\sigma} \\ &\qquad -(A_{ijk}-1)\varepsilon^k\ \epsilon^{\mu\nu\rho\sigma}\ F^i_{\mu\nu}F^j_{\rho\sigma}\Big\}\ . \end{aligned} \quad \text{(A.13)}$$

We will now fix the symmetric scheme $A_{ijk} = -B_{ijk} = 1/3$ in which we obtain the gauge variation

$$\delta S_{ijk} \ = \ -\frac{t_{ijk}}{3!(12\pi^2)}\int d^4x\ \Big\{\varepsilon^i\ \ F^j\wedge F^k + \varepsilon^j\ \ F^i\wedge F^k + \varepsilon^k\ \ F^i\wedge F^j\Big\} \quad \text{(A.14)}$$

where we used

$$F^i\wedge F^j = \frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\ F^i_{\mu\nu}F^j_{\rho\sigma} \quad \text{(A.15)}$$

We now sum over the U(1)'s to obtain the full cubic effective action in the symmetric scheme. Its gauge variation is

$$\delta S_3 = \sum_{i,j,k}\delta S_{ijk} = -\frac{t_{ijk}}{24\pi^2}\int d^4x\ \varepsilon^i F^j\wedge F^k \quad \text{(A.16)}$$

where we have reinstated our summation convention.

We may now study the effect of changing the scheme of the triangle graphs. This is obtained by setting

$$A_{ijk} = \frac{1}{3} + \tilde{A}_{ijk} \quad , \quad B_{ijk} = -\frac{1}{3} + \tilde{B}_{ijk} \tag{A.17}$$

The gauge variation now becomes

$$\delta S_3 = -\frac{t_{ijk}}{24\pi^2} \int d^4x \, \left\{ \varepsilon^i \quad F^j \wedge F^k \right\} \tag{A.18}$$
$$-\frac{t_{ijk}}{3!(8\pi^2)} \int d^4x \Big\{ \tilde{A}_{ijk}(\varepsilon^i F^j \wedge F^k - \varepsilon^k F^i \wedge F^j) - \tilde{B}_{ijk}(\varepsilon^i F^j \wedge F^k - \varepsilon^j F^i \wedge F^k) \Big\}$$

The extra terms have the same transformation properties as

$$S_{\text{counter}} = -\frac{t_{ijk}}{3!(16\pi^2)} \int d^4x \; \epsilon^{\mu\nu\rho\sigma} \left[\tilde{A}_{ijk} \; A^i_\mu A^k_\nu F^j_{\rho\sigma} - \tilde{B}_{ijk} \; A^i_\mu A^j_\nu F^k_{\rho\sigma} \right] \tag{A.19}$$

Therefore, in this new scheme, the new effective action is obtained from the old one by adding the GCS terms in (A.19).

A more direct way to see this is to compute the variation of the effective action between two different regularisation schemes specified by the shift vectors a_1^{ijk} and a_2^{ijk}, where $a^{ijk} = A_{ijk} k_1 + B_{ijk} k_2$:

$$\Delta\Gamma^{\mu\nu\rho}_{ijk}(x,y,z) = \Gamma^{\mu\nu\rho}_{ijk}|_{a_1} - \Gamma^{\mu\nu\rho}_{ijk}|_{a_2} \; . \tag{A.20}$$

By Taylor expanding

$$\Delta\Gamma^{\mu\nu\rho}(p,k_i,a) = (a_2 - a_1)^\sigma \frac{\partial}{\partial a_1^\sigma} \Gamma^{\mu\nu\rho}(p,k_i,a_1)$$
$$+\frac{1}{2}(a_2 - a_1)^{\sigma_1}(a_2 - a_1)^{\sigma_2} \frac{\partial^2}{\partial a_1^{\sigma_1} \partial a_2^{\sigma_2}} \Gamma^{\mu\nu\rho}(k_i,a_1) + \cdots \tag{A.21}$$

and noticing that $\partial\Gamma^{\mu\nu\rho}(p,k_i,a)/\partial a^\sigma \partial\Gamma^{\mu\nu\rho}(p,k_i,a)/\partial p^\sigma$, we can cast the scheme difference into the form

$$\Delta\Gamma^{\mu\nu\rho}_{ijk}(x,y,z) = \frac{i}{(2\pi)^{12}} \int d^4k_1 d^4k_2 \; e^{-i(k_1+k_2)x + ik_1 y + ik_2 z} (a_2 - a_1)^\sigma \times$$
$$\int d^4p \; t_{ijk} \frac{\partial}{\partial p^\sigma} \left[\Gamma^{\nu\rho\mu}_{ijk}(p,k_1,k_2,a_1) - \Gamma^{\rho\nu\mu}_{ijk}(p,k_2,k_1,-a_1) \right] \; + \cdots \; , \tag{A.22}$$

where $\cdots$ are contributions at least quadratic in the shift vectors a containing at least second derivatives with respect to the loop momentum p. Since all contributions come from the boundary of the loop momentum space, we will see in a moment that only the first contribution gives a non-vanishing contribution. Like in the case of the triangle gauge anomalies, the quantity $\Delta\Gamma^{\mu\nu\rho}_{ijk}$ is given by a surface contribution. A simple counting of the leading momentum dependence for $p \to \infty$ shows that only the leading contribution

$$\Gamma^{\nu\rho\mu}_{ijk}(p,k_1,k_2,a) \to -\frac{2}{p^6} \left[p^2(p^\mu \eta^{\nu\rho} + p^\nu \eta^{\mu\rho} + p^\rho \eta^{\mu\nu}) - 4p^\mu p^\nu p^\rho + ip^2 \epsilon^{\nu\rho\mu\sigma} p_\sigma \right] \tag{A.23}$$

is giving a non-vanishing result and only the last term in (A.23) contributes to (A.22). By explicitly computing now the surface integral

$$\int d^4p\ \partial_\sigma \frac{p_\epsilon}{p^4} = -\frac{1}{8}\eta_{\sigma\epsilon} \int d^4p\ \partial^2 \frac{1}{p^2} = -\frac{\pi^2}{4}\eta_{\sigma\epsilon}\ , \tag{A.24}$$

we finally get the difference of the effective action in two different regularisation schemes to be equal to

$$\begin{aligned} \Delta S_3^{\rm an} &= \frac{1}{3!}\int d^4x\ d^4y\ d^4z\ \Delta\Gamma^{\mu\nu\rho}_{ijk}(x,y,z)\ A^i_\mu(x)\ A^j_\nu(y)\ A^k_\rho(z) \\ &= \frac{1}{32\pi^2} t_{ijk}\ \left(\Delta A_{ikj} - \Delta B_{ijk}\right) \int\ A^i \wedge A^j \wedge F^k\ . \end{aligned} \tag{A.25}$$

We will do hear a counting of the relevant parameters. we start with all possible independent GCS terms $\mathcal{S}_{ijk}$ constructed out of N abelian gauge bosons. The relations are, antisymmetry in the first two indices as well as cyclic symmetry

$$\mathcal{S}_{ijk} + \mathcal{S}_{jik} = 0 \quad , \quad \mathcal{S}_{ijk} + \mathcal{S}_{jki} + \mathcal{S}_{kij} = 0 \tag{A.26}$$

We have to distinguish the following cases:

iii) Then the GCS term is trivial

ijj) $i \neq j$. There are two possible GCS terms per pair of distinct gauge bosons, namely $\mathcal{S}_{ijj}$ and $\mathcal{S}_{jii}$. This gives a total of $N(N-1)$ independent terms.

ijk) with $i \neq j \neq k \neq i$. Here out of the three possible terms only two are independent. The third is related to the other two by the cyclicity property in (A.26). We therefore obtain here $\frac{N(N-1)(N-2)}{3}$ independent terms.

Therefore the total number is $\frac{N(N^2-1)}{3}$ corresponding to the Young tableau $\yng(2,1)$.

It would naively seem that this number is smaller than the number of possible schemes, specified by the coefficients $\tilde{A}_{ijk}$, $\tilde{B}_{ijk}$, namely , $2N^3$. We will now show that the number of relevant scheme parameters is exactly equal to the number of independent GCS terms.

iii) For this case, we must choose $\tilde{A}_{iii} = \tilde{B}_{iii} = 0$ to respect the full Bose symmetry of the triangle graph.

iij) $i \neq j$. In this case, (A.18) indicates that the scheme depends only on $\tilde{A} - \tilde{B}$, and there are N(N-1) such coefficients.

ijk) with $i \neq j \neq k \neq i$. For this case we have $2 \times \frac{N(N-1)(N-2)}{3!}$ such coefficients.

Therefore the scheme dependence of triangle graphs is in one to one correspondence with all possible GCS terms.

We now split the U(1)s into two groups, as was done in section 2. We also add the non-abelian mixed graphs. Using (A.14) we obtain the gauge variation of the effective action due to the triangle graphs in the symmetric scheme,

$$\begin{aligned} \delta S_{\rm triangle} = -\frac{1}{24\pi^2}\int \Big\{ & t_{abc}\varepsilon^a\ F^b \wedge F^c + t_{mnr}\varepsilon^m\ F^n \wedge F^r \\ & + t_{mab}(2\varepsilon^a\ F^b \wedge F^m + \varepsilon^m\ F^a \wedge F^b) \\ & + t_{amn}(2\varepsilon^m\ F^a \wedge F^n + \varepsilon^a\ F^m \wedge F^n) \\ & + T^a{}_\alpha(2{\rm Tr}[\varepsilon\ \tilde{G}^\alpha] \wedge F^a + \varepsilon^a\ {\rm Tr}[G^\alpha \wedge G^\alpha]) \\ & + T^m{}_\alpha(2{\rm Tr}[\varepsilon\ \tilde{G}^\alpha] \wedge F^m + \varepsilon^m\ {\rm Tr}[G^\alpha \wedge G^\alpha])\Big\} \end{aligned} \tag{A.27}$$

The tensor T is given by the cubic traces of the U(1) and non-abelian generators,

$$T^a{}_\alpha = \mathrm{Tr}[\mathcal{Q}_a (TT)^\alpha] \quad , \quad T^m{}_\alpha = \mathrm{Tr}[\mathcal{Q}_m (TT)^\alpha] \tag{A.28}$$

with $(TT)^\alpha$ the quadratic Casimir of the α-th non-abelian factor.

B. Basis changes

In this appendix we relate effective couplings in the D-brane basis, as calculated in string theory and the diagonal basis, where the gauge-boson mass-matrix is diagonal. This basis was introduced in detail in section 2, (2.4)-(2.11). We obtain the following equations relating the PQ couplings

$$C^M{}_{ab} = W^I_M \eta^a_i \eta^b_j C^I{}_{ij} \quad , \quad C^M{}_{mn} = W^I_M \eta^m_i \eta^n_j C^I{}_{ij} \quad , \quad C^M{}_{am} = 2W^I_M \; \eta^a_i \eta^m_j \; C^I{}_{ij} \tag{B.1}$$

$$C^a{}_{bc} = W^I_a M_a \eta^b_i \eta^c_j C^I{}_{ij} \quad , \quad C^a{}_{mn} = W^I_a M_a \eta^m_i \eta^n_j C^I{}_{ij} \quad , \quad C^a{}_{bm} = 2W^I_a M_a \; \eta^b_i \eta^m_j \; C^I{}_{ij} \tag{B.2}$$

$$D^M{}_\alpha = W^I_M \; C^I{}_\alpha \quad , \quad D^a{}_\alpha = W^I_a \; M_a \; C^I{}_\alpha \tag{B.3}$$

$$\begin{aligned} C^I{}_{ij} = \frac{M^I_k \eta^a_k}{M^2_a} &\left[C^a{}_{bc} \eta^b_i \eta^c_j + \frac{1}{2} C^a{}_{bm} (\eta^m_i \eta^b_j + \eta^m_j \eta^b_i) + C^a{}_{mn} \eta^m_i \eta^n_j \right] + \\ + W^I_M &\left[C^M{}_{bc} \eta^b_i \eta^c_j + \frac{1}{2} C^M{}_{mb} (\eta^m_i \eta^b_j + \eta^m_j \eta^b_i) + C^M{}_{mn} \eta^m_i \eta^n_j \right] \end{aligned} \tag{B.4}$$

$$M^I_i C^I{}_{jk} = \eta^a_i \left[C^a{}_{bc} \eta^b_j \eta^c_k + \frac{1}{2} C^a{}_{bm} (\eta^m_j \eta^b_k + \eta^m_k \eta^b_j) + C^a{}_{mn} \eta^m_j \eta^n_k \right] \tag{B.5}$$

where we used

$$M^I_i W^I_M = 0 \tag{B.6}$$

the GCS couplings

$$E_{abc} = \eta^a_i \eta^b_j \eta^c_k \; E_{ijk} \quad , \quad E_{mnr} = \eta^m_i \eta^n_j \eta^r_k \; E_{ijk} \tag{B.7}$$

$$E_{man} = 2(\eta^m_i \eta^a_j \eta^n_k + \eta^m_i \eta^n_j \eta^a_k) E_{ijk} \tag{B.8}$$

$$E_{mab} = 2(\eta^m_i \eta^a_j \eta^b_k - \eta^a_i \eta^b_j \eta^m_k) E_{ijk} \tag{B.9}$$

$$Z^a{}_\alpha = \eta^a_i \; Z^i{}_\alpha \quad , \quad Z^m{}_\alpha = \eta^m_i \; Z^i{}_\alpha \tag{B.10}$$

$$E_{ijk} = \eta^m_i \eta^n_j \eta^r_k \; E_{mnr} + \frac{1}{2}(\eta^m_i \eta^a_j - \eta^m_j \eta^a_i) \eta^n_k \; E_{man} + \frac{1}{2}(\eta^m_i \eta^a_j - \eta^m_j \eta^a_i) \eta^b_k \; E_{mab} + \eta^a_i \eta^b_j \eta^c_k \; E_{abc} \tag{B.11}$$

For the charges, we start from the coupling

$$S_{minimal} = \int \; \bar\psi \; \mathcal{Q}_i \; A^i_\mu \gamma^\mu \; \psi \tag{B.12}$$

and by changing basis this becomes

$$S_{minimal} = \int \; \left[\bar\psi \; \mathcal{Q}_a \; Q^a_\mu \gamma^\mu \; \psi + \bar\psi \; \mathcal{Q}_m \; Y^m_\mu \gamma^\mu \; \psi \right] \tag{B.13}$$

Twist Group / Gauge Group	(99)/(55) matter	(95) matter
Z_6	$2(15,1,1)+2(1,\overline{15},1)$	$(6,1,1;6,1,1)+(1,\overline{6},1;1,\overline{6},1)$
$U(6)_9^2\times U(4)_9\times$	$+2(\overline{6},1,4)+2(1,6,\overline{4})$	$+(1,6,1;1,1,\overline{4})+(1,1,\overline{4};1,6,1)$
$U(6)_5^2\times U(4)_5$	$+(\overline{6},1,\overline{4})+(1,6,4)+(6,\overline{6},1)$	$+(\overline{6},1,1;1,1,4)+(1,1,4;\overline{6},1,1)$
Z_6'	$(\bar{4},1,8)+(1,4,\bar{8})+(6,1,1)+(1,\bar{6},1)$	$(\bar{4},1,1;\bar{4},1,1)+(1,4,1;1,4,1)$
$U(4)_9^2\times U(8)_9\times$	$+(4,1,8)+(1,\bar{4},\bar{8})+(\bar{4},4,1)+(1,1,28)$	$+(1,\bar{4},1;1,1,8)+(1,1,\bar{8};1,\bar{4},1)$
$U(4)_5^2\times U(8)_5$	$+(1,1,\overline{28})+(4,4,1)+(\bar{4},\bar{4},1)$	$+(4,1,1;1,1,\overline{8})+(1,1,\overline{8};4,1,1)$

Table 1: The transformations of the massless fermionic states in the Z_6 and Z_6' D=4 orientifold.

with

$$\mathcal{Q}_a=\eta_i^a\mathcal{Q}_i \quad , \quad \mathcal{Q}_m\eta_i^m\mathcal{Q}_i \tag{B.14}$$

Therefore

$$t_{abc}=\eta_i^a\eta_j^b\eta_k^c\; t_{ijk} \quad , \quad t_{abm}=\eta_i^a\eta_j^b\eta_k^m\; t_{ijk} \quad , \quad t_{amn}\eta_i^a\eta_j^m\eta_k^n\; t_{ijk} \tag{B.15}$$

It is also convenient to introduce the projections

$$G^{ij}\equiv\eta_i^a\eta_j^a \quad , \quad \tilde{G}^{ij}\equiv\eta_i^m\eta_j^m \quad , \quad G^{ij}+\tilde{G}^{ij}=\delta^{ij} \tag{B.16}$$

G projects on the subspace of massive U(1)s while $\tilde{G}$ to the massless one,

$$G^{ij}G^{jk}=G^{ik} \quad , \quad \tilde{G}^{ij}\tilde{G}^{jk}=\tilde{G}^{ik} \quad , \quad \tilde{G}^{ij}G^{jk}=G^{ij}\tilde{G}^{jk}=0 \tag{B.17}$$

We also define

$$\tilde{M}_{ij}\equiv\frac{1}{M_a^2}\eta_i^a\eta_j^a \tag{B.18}$$

This is the inverse of M_{ij}^2 in the invertible (massive) subspace. It satisfies

$$\tilde{M}_{ij}\eta_j^a=\frac{1}{M_a^2}\eta_i^a \quad , \quad \tilde{M}_{ij}\eta_j^m=0 \tag{B.19}$$

C. Explicit Orientifold Examples

Here we evaluate explicitly the anomaly-related charge traces and the coefficients of the generalized Chern-Simons terms for the Z_6 and Z_6' orientifolds. We have chosen these examples as they contain all the non-trivial ingredients of a generic orientifold vacuum.

C.1 Z_6 orbifold

The orbifold rotation vector is $(v_1,v_2,v_3)=(1,1,-2)/6$. There is an order two twist ($k=3$) and we must have one set of D5-branes. Tadpole cancellation then implies the existence of 32 D9-branes and 32 D5-branes, that we put together at the origin of the internal space. The Chan–Paton vectors are

$$V_9=V_5=\frac{1}{12}(1,1,1,1,1,1,5,5,5,5,5,5,3,3,3,3)\ , \tag{C.1}$$

giving

$$tr[\gamma_k] = 0 \quad \text{for} \quad k = 1,3,5 \quad , \quad tr[\gamma_2] = 4 \quad , \quad tr[\gamma_4] = -4 \,. \tag{C.2}$$

The gauge group has a factor of $U(6) \times U(6) \times U(4)$ coming from the D9-branes and an isomorphic factor coming from the D5-branes. The massless spectrum is provided in table 1. The $\mathcal{N} = 1$ sectors correspond to $k = 1,2,4,5$, while $k = 3$ is an $\mathcal{N} = 2$ sector.

Anomaly traces

Here we evaluate the mixed anomaly matrixes, from the massless spectrum of the Z_6 orientifold (table 1). We normalize the generators of $U(1)_i$ so that the charges are ± 1 for fundamentals and ± 2 for symmetric / antisymmetric tensors. This implies that $Q_{1,2} = \lambda_{1,2}$ while $Q_3 = \frac{2}{\sqrt{6}}\lambda_3$ where λ_i are the generators given in (3.6).

We also normalize the generators of the non-abelian factors as $Tr[T_i T_j]_{\square} = \delta_{ij}$ in the fundamental. This implies for example that for $SU(N)$, the same trace gives $Tr[T_i T_j]_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} = (N-2)\delta_{ij}$ for the antisymmetric representation.

Therefore, for the mixed anomalies between abelian and non-abelian factors we have:

$$t_{ia} \equiv Tr[Q_i (T^A T^A)_a] = \begin{pmatrix} 6 & -3 & 2 & 3 & 0 & 2 \\ 3 & -6 & -2 & 0 & -3 & -2 \\ -9 & 9 & 0 & -3 & 3 & 0 \\ 3 & 0 & 2 & 6 & -3 & 2 \\ 0 & -3 & -2 & 3 & -6 & -2 \\ -3 & 3 & 0 & -9 & 9 & 0 \end{pmatrix} \tag{C.3}$$

where the columns label the U(1)s and the rows the non-abelian factors. It can be directly verified that there are three linear combinations of the U(1)s which are free of 4d mixed non-abelian anomalies.

For abelian mixed anomalies we must evaluate $t_{ijk} = Tr[Q_i Q_j Q_k]$. It is enough to calculate

$$t_{ij} = \begin{cases} t_{ijj} & , \text{ for } i \neq j \\ 3t_{ijj} & , \text{ for } i = j \end{cases} \tag{C.4}$$

because in this basis (D-brane basis), $t_{ijk} = 0$ when all i,j,k are distinct (chiral fermions can carry at most two U(1) charges). Therefore:

$$t_{ij} = \begin{pmatrix} 216 & -36 & 24 & 36 & 0 & 24 \\ 36 & -216 & -24 & 0 & -36 & -24 \\ -72 & 72 & 0 & -24 & 24 & 0 \\ 36 & 0 & 24 & 216 & -36 & 24 \\ 0 & -36 & -24 & 36 & -216 & -24 \\ -24 & 24 & 0 & -72 & 72 & 0 \end{pmatrix} \tag{C.5}$$

Anomalous U(1) masses

The various contributions to the mass matrix are

$$\frac{1}{2}M^2_{99,ij} = -\frac{\sqrt{3}}{48\pi^3}\Big(tr[\gamma_1\lambda^9_i]tr[\gamma_1\lambda^9_j] + tr[\gamma_5\lambda^9_i]tr[\gamma_5\lambda^9_j] \\ +3(tr[\gamma_2\lambda^9_i]tr[\gamma_2\lambda^9_j] + tr[\gamma_4\lambda^9_i]tr[\gamma_4\lambda^9_j])\Big) - \frac{\mathcal{V}_3}{3\pi^3}tr[\gamma_3\lambda^9_i]tr[\gamma_3\lambda^9_j] \tag{C.6}$$

and similarly for $M_{55,ij}$, while

$$\frac{1}{2}M^2_{95,ij} = -\frac{\sqrt{3}}{48\pi^3}\Big([tr[\gamma_1\lambda^9_i]tr[\gamma_1\lambda^5_j] + tr[\gamma_5\lambda^9_i]tr[\gamma_5\lambda^5_j] \\ +tr[\gamma_2\lambda^9_i]tr[\gamma_2\lambda^5_j] + tr[\gamma_4\lambda^9_i]tr[\gamma_4\lambda^5_j]\Big) - \frac{\mathcal{V}_3}{12\pi^3}tr[\gamma_3\lambda^9_i]tr[\gamma_3\lambda^5_j]\,. \tag{C.7}$$

This mass matrix has the following eigenvalues and eigenvectors:

$$\begin{array}{ll} m_1^2 = 0 & , \quad A_1 + A_2 - \tilde{A}_1 - \tilde{A}_2 + \sqrt{6}(A_3 - \tilde{A}_3), \\ m_2^2 = 3\sqrt{3}/2 & , \quad A_1 - A_2 - \tilde{A}_1 + \tilde{A}_2, \\ m_3^2 = 3\sqrt{3} & , \quad A_1 - A_2 + \tilde{A}_1 - \tilde{A}_2, \\ m_4^2 = 40\mathcal{V}_3/3 & , \quad -\sqrt{\tfrac{3}{2}}(A_1 + A_2 - \tilde{A}_1 - \tilde{A}_2) - A_3 + \tilde{A}_3, \\ m_\pm^2 = \frac{7\sqrt{3}+80\mathcal{V}_3\pm\sqrt{147-1040\sqrt{3}\mathcal{V}_3+6400\mathcal{V}_3^2}}{12} & , \quad a_\pm(A_1 + A_2 + \tilde{A}_1 + \tilde{A}_2) + A_3 + \tilde{A}_3 \end{array} \tag{C.8}$$

where $A, \tilde{A}$ denote the abelian bosons which are coming from D9 and D5 branes respectively. Also

$$a_\pm = \frac{40\mathcal{V}_3 - \sqrt{3} \pm \sqrt{147 - 1040\sqrt{3}\mathcal{V}_3 + 6400\mathcal{V}_3^2}}{12\sqrt{2} - 40\sqrt{6}\mathcal{V}_3}\,. \tag{C.9}$$

In the limit $\mathcal{V}_3 \to 0$ two more masses become zero (m_4 and m_-). It is straightforward to check that these three gauge are anomaly-free in four bosons dimensions. This behavior, was explained in detail in [14].

Generalized CS terms from the non-planar cylinder

Using formulae (C.6-C.7), we can evaluate the couplings of axions to one and two gauge bosons, (M_I's and C_I's respectively). Since axions (which are coming from the twisted closed string sector) are localized at the fixed points, it is necessary to identify the fixed points in each sector and their properties. In figure 2 we denote the fixed points on each torus under the Z_6 action.

The $k = 1$ sector provides 3 points fixed under the Z_6 action. The $k = 2$ sector provides 27 points fixed under the Z_3 action. However, the Z_6 action leaves invariant only 3 of them (the ones of the $k = 1$ sector) and relates doublets of the rest. In total there are 12 doublets of points which are identified under the $Z_6/Z_3 = Z_2$ action. The $k = 3$ sector provides 16 points fixed under the Z_2 action. However, the Z_6 action leaves invariant only 1 of them (which is located at the origin) and relates triplets of the rest. In total there are 5 triplets of points identified under the $Z_6/Z_2 = Z_3$ action.

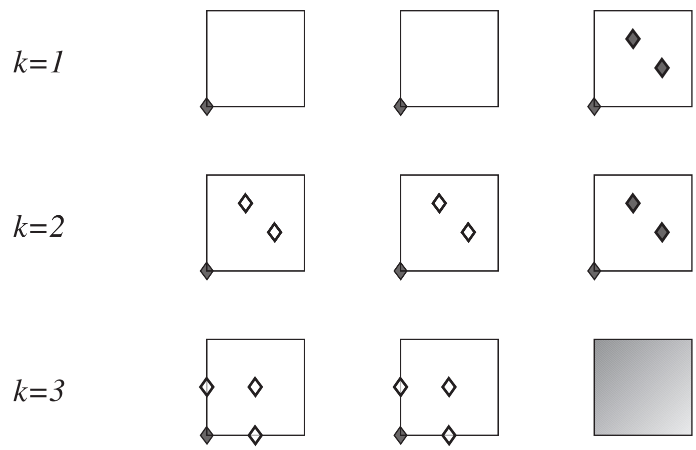


Figure 2: We denote by ♦/◊ the fixed points on each torus, which are invariant/related to others by the Z_6 action.

Taking all this into account, the D9 branes couple to axions as:

$$
\begin{aligned}
M^{1,\circlearrowright}_{a(9)} &= \frac{i}{\sqrt{48\pi^3}}\frac{1}{3^{1/4}}tr[\gamma_1\lambda^9_a] \\
M^{2,\circlearrowright}_{a(9)} &= \frac{i}{\sqrt{48\pi^3}}\frac{1}{27^{1/4}}tr[\gamma_2\lambda^9_a]\ , \qquad M^{2,\rightleftarrows}_{a(9)} = \frac{i}{\sqrt{48\pi^3}}\frac{\sqrt{2}}{27^{1/4}}tr[\gamma_2\lambda^9_a] \\
M^{3,\circlearrowright}_{a(9)} &= \frac{i\sqrt{2\mathcal{V}_3}}{\sqrt{24\pi^3}}\frac{1}{16^{1/4}}tr[\gamma_3\lambda^9_a]\ , \qquad M^{3,\rightleftarrows}_{a(9)} = \frac{i\sqrt{2\mathcal{V}_3}}{\sqrt{24\pi^3}}\frac{\sqrt{3}}{16^{1/4}}tr[\gamma_3\lambda^9_a]
\end{aligned}
\tag{C.10}
$$

where $\circlearrowright$ denotes fixed points of the kth sector which are also fixed under the larger Z_6 orbifold action (corresponding to ♦ on all two-tori T^2_i) and $\rightleftarrows$ denotes fixed points which are related by the larger Z_6 orbifold action (with ◊ in at least one tori T^2_i) .

If all D5 branes are at the origin then the corresponding couplings are:

$$
\begin{aligned}
M^{1,\text{origin}}_{a(5)} &= \frac{i}{\sqrt{48\pi^3}}\frac{1}{3^{1/4}}tr[\gamma_1\lambda^5_a] \\
M^{2,\text{origin}}_{a(5)} &= \frac{i}{\sqrt{48\pi^3}}3^{1/4}tr[\gamma_2\lambda^5_a] \\
M^{3,\text{origin}}_{a(5)} &= \frac{i\sqrt{2\mathcal{V}_3}}{\sqrt{24\pi^3}}16^{1/4}tr[\gamma_3\lambda^5_a]
\end{aligned}
\tag{C.11}
$$

The coefficients of C_Is are proportional to the coefficients of M_Is (without the traces):

$$
C^{k=1}_{(99,55)} = -4M^1_{(9,5)}\ , \quad C^{k=2}_{(99,55)} = -4M^2_{(9,5)}\ , \quad C^{k=3}_{(99,55)} = -M^3_{(9,5)}\ . \tag{C.12}
$$

for the various sectors in (C.10,C.11).

Now, we can evaluate the symmetric tensor t_{ijl} for the Z_6 orientifold using:

$$
t^{Z_N}_{ijl} = \sum_{k=1}^{N-1}\sum_f \eta_k\left(M^{k,f}_i C^{k,f}_{jl} + M^{k,f}_j C^{k,f}_{li} + M^{k,f}_l C^{k,f}_{ij}\right)\ , \tag{C.13}
$$

where for axionic exchange between D9-D9 and D5-D5 $\eta_1 = \eta_2 = -\eta_4 = -\eta_5 = -1$ however, between D9-D5 $\eta_1 = \eta_2 = -\eta_4 = -\eta_5 = 1$. In all cases $\eta_3 = 0$. M_Is and C_Is are given in (C.10, C.11, C.12).

Using unnormalized λ's ((3.6) without the coefficient $1/2\sqrt{n_i}$ that normalizes $Tr[\lambda_i\lambda_j] = \delta_{ij}/2$) we find perfect agreement with the anomaly matrixes t_{ijk} of the previous section (C.5). We stress that this equation holds irrespective of the scheme used in calculating triangle graphs in the effective field theory.

We now evaluate the antisymmetric combination

$$E^{Z_N}_{ijl} = \sum_{k=1}^{N-1}\sum_f \eta_k \left(M_i^{k,f} C_{jl}^{k,f} - M_j^{k,f} C_{li}^{k,f}\right) \tag{C.14}$$

that provides the coefficients of the GCS terms, and we find it is non-zero. We focus on elements $E_{ijj} = -E_{jij}$ since all other vanish, $E_{iij} = E_{ijl} = 0$:

$$E_{ij} = E_{ijj} = -E_{jij} = \begin{pmatrix} 0 & 36 & -72 & 36 & 0 & -24 \\ -36 & 0 & 72 & 0 & -36 & 24 \\ 24 & -24 & 0 & 24 & -24 & 0 \\ 36 & 0 & -24 & 0 & 36 & -72 \\ 0 & -36 & 24 & -36 & 0 & 72 \\ 24 & -24 & 0 & 24 & -24 & 0 \end{pmatrix} . \tag{C.15}$$

Therefore, in the natural EFT regularization scheme which treats democratically the anomalous currents, we need GCS terms to cancel the anomalies in the Z_6 orientifold.

C.2 Z_6' orbifold

The orbifold rotation vector is $(v_1, v_2, v_3) = (1, -3, 2)/6$. There is an order two twist ($k = 3$) and we must have one set of D5-branes. Tadpole cancellation then implies the existence of 32 D9-branes and 32 D5-branes, as in the previous example, that we put together at the origin of the internal space. The Chan–Paton vectors are

$$V_9 = V_5 = \frac{1}{12}(1,1,1,1,5,5,5,5,3,3,3,3,3,3,3,3) \ , \tag{C.16}$$

implying

$$tr[\gamma_k] = 0 \quad \text{for} \quad k = 1,3,5 \quad , \quad tr[\gamma_2] = -8 \quad , \quad tr[\gamma_4] = 8\,. \tag{C.17}$$

The gauge group has a factor of $U(4)\times U(4)\times U(8)$ coming from the D9-branes and an isomorphic factor coming from the D5-branes. The massless spectrum is provided in table 1. The $\mathcal{N} = 1$ sectors correspond to $k = 1, 5$, while $k = 2, 3, 4$ are $\mathcal{N} = 2$ sectors.

Anomaly traces

Normalizing the generators as for the Z_6 case, we have for the mixed anomalies between abelian and non-abelian factors:

$$t_{ia} \equiv Tr[Q_i(T^A T^A)_a] = \begin{pmatrix} 2 & 2 & 8 & -2 & 0 & -4 \\ -2 & -2 & -8 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ -2 & 0 & -4 & 2 & 2 & 8 \\ 0 & 2 & 4 & -2 & -2 & -8 \\ 2 & -2 & 0 & 0 & 0 & 0 \end{pmatrix} \,, \tag{C.18}$$

where the columns label again the U(1)s and the rows the non-abelian factors. We also evaluate the mixed anomalies of abelian factors $t_{ijk} = Tr[Q_i Q_j Q_k]$ (here we provide again the t_{ij} (C.4)).

$$t_{ij} = \begin{pmatrix} 48 & 16 & 64 & -16 & 0 & -32 \\ -16 & -48 & -64 & 0 & 16 & 32 \\ 0 & 0 & 0 & 32 & -32 & 0 \\ -16 & 0 & -32 & 48 & 16 & 64 \\ 0 & 16 & 32 & -16 & -48 & -64 \\ 32 & -32 & 0 & 0 & 0 & 0 \end{pmatrix} \,. \tag{C.19}$$

In total, there are six U(1)s. Four of them are anomalous and two are free of 4d anomalies.

Anomalous U(1) masses

The contribution to the mass matrix is:

$$\begin{aligned} \frac{1}{2}M^2_{aa,ij} &= -\frac{\sqrt{3}}{24\pi^3}\left(tr[\gamma_1\lambda_i]tr[\gamma_1\lambda_j] + tr[\gamma_5\lambda_i]tr[\gamma_5\lambda_j]\right) \\ &\quad - \frac{(2\mathcal{V}_2)^{\epsilon_a}}{8\pi^3}\left(tr[\gamma_2\lambda_i]tr[\gamma_2\lambda_j] + tr[\gamma_4\lambda_i]tr[\gamma_4\lambda_j]\right) - \frac{\mathcal{V}_3}{3\pi^3}tr[\gamma_3\lambda_i]tr[\gamma_3\lambda_j] \end{aligned} \tag{C.20}$$

where $a = 9, 5$ and $\epsilon_{9,5} = \pm 1$ respectively, while

$$\begin{aligned} \frac{1}{2}M^2_{95,ij} = &- \frac{\sqrt{3}}{48\pi^3}\Big(tr[\gamma_1\lambda_i]tr[\gamma_1\tilde{\lambda}_j] + tr[\gamma_5\lambda_i]tr[\gamma_5\tilde{\lambda}_j] \\ &+ tr[\gamma_2\lambda_i]tr[\gamma_2\tilde{\lambda}_j] - tr[\gamma_4\lambda_i]tr[\gamma_4\tilde{\lambda}_j]\Big) - \frac{\mathcal{V}_3}{12\pi^3}tr[\gamma_3\lambda_i]tr[\gamma_3\tilde{\lambda}_j] \end{aligned} \tag{C.21}$$

Thus, the unormalized mass matrix has eigenvalues and eigenvectors:

$$\begin{array}{lcl} m_1^2 = 6\mathcal{V}_2 & , & -A_1 + A_2, \\ m_2^2 = 3/(2\mathcal{V}_2) & , & -\tilde{A}_1 + \tilde{A}_2, \\ m_{3,4}^2 = \frac{5\sqrt{3}+48\mathcal{V}_3 \pm\sqrt{3(25-128\sqrt{3}\mathcal{V}_3+768\mathcal{V}_3^2)}}{12} & , & \pm a_\pm(A_1 + A_2 - \tilde{A}_1 - \tilde{A}_2) - A_3 + \tilde{A}_3, \\ m_{5,6}^2 = \frac{15\sqrt{3}+80\mathcal{V}_3 \pm\sqrt{5(135-384\sqrt{3}\mathcal{V}_3+1280\mathcal{V}_3^2)}}{12} & , & b_\pm(A_1 + A_2 + \tilde{A}_1 + \tilde{A}_2) + A_3 + \tilde{A}_3 \end{array} \tag{C.22}$$

where

$$a_\pm = \frac{\mp 3 + \sqrt{25 - 128\sqrt{3}\mathcal{V}_3 + 768\mathcal{V}_3^2}}{4\sqrt{2}(4\sqrt{3}\mathcal{V}_3 - 1)} \quad , \qquad b_\pm = \frac{\pm 9\sqrt{3} - \sqrt{5(135 - 384\sqrt{3}\mathcal{V}_3 + 1280\mathcal{V}_3^2)}}{4\sqrt{2}(20\mathcal{V}_3 - 3\sqrt{3})} \,. \tag{C.23}$$

Note that the eigenvalues are always positive. They are also invariant under the T-duality symmetry of the theory $\mathcal{V}_2 \to 1/4\mathcal{V}_2$. Thus, all U(1)s become massive, including the two anomaly free combinations. The reason is that these combinations are anomalous in six dimensions. Observe however that in the limit $\mathcal{V}_3 \to 0$, the two linear combinations that are free of four-dimensional anomalies become massless. This is consistent with the fact that the six-dimensional anomalies responsible for their mass cancel locally in this limit [14].

Generalized Chern-Simons terms

As for the Z_6 case, we identify the fixed points and the couplings of the axions to the branes.

The $k = 1$ sector provides 12 points fixed under the Z_6' action. The $k = 2$ sector provides 9 points fixed under the Z_3 action. However, the Z_6' action leaves invariant only 3 of them and relates doublets of the rest. In total there are 3 doublets of points which are identified under the $Z_6'/Z_3 = Z_2$ action. The $k = 3$ sector provides 16 points fixed under the Z_2 action. However, the Z_6' action leaves invariant only 4 of them and relates triplets of the rest. In total there are 4 triplets of points identified under the $Z_6'/Z_2 = Z_3$ action. In figure 3 we denote the fixed points under the Z_6' action.

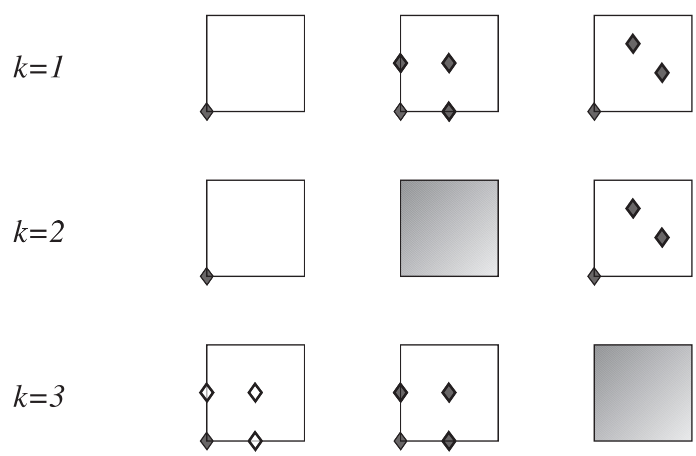


Figure 3: We denote by $\blacklozenge/\lozenge$ the fixed points on each torus, which are invariant/related to others by the Z_6' action.

Therefore here the D9 branes couple to axions with:

$$
\begin{aligned}
M^{1,\circlearrowleft}_{a(9)} &= \frac{i}{\sqrt{48\pi^3}}\frac{1}{12^{1/4}}tr[\gamma_1\lambda_a] \ , \\
M^{2,\circlearrowleft}_{a(9)} &= \frac{i\sqrt{2\mathcal{V}_2}}{\sqrt{24\pi^3}}\frac{1}{9^{1/4}}tr[\gamma_2\lambda_a] \ , \qquad M^{2,\rightleftarrows}_{a(9)} = \frac{i\sqrt{2\mathcal{V}_2}}{\sqrt{24\pi^3}}\frac{\sqrt{2}}{9^{1/4}}tr[\gamma_2\lambda_a] \ , \\
M^{3,\circlearrowleft}_{a(9)} &= \frac{i\sqrt{2\mathcal{V}_3}}{\sqrt{24\pi^3}}\frac{1}{16^{1/4}}tr[\gamma_3\lambda_a] \ , \qquad M^{3,\rightleftarrows}_{a(9)} = \frac{i\sqrt{2\mathcal{V}_3}}{\sqrt{24\pi^3}}\frac{\sqrt{3}}{16^{1/4}}tr[\gamma_3\lambda_a] \ ,
\end{aligned} \tag{C.24}
$$

where again $\circlearrowleft$ denotes fixed points of the kth sector which are also fixed under the larger Z_6' orbifold action and $\rightleftarrows$ denotes fixed points which are related to others by the larger Z_6' orbifold action.

If all D5 branes are at the origin,

$$
\begin{aligned}
M_{a(5)}^{1,\text{origin}} &= \frac{i}{\sqrt{48\pi^3}}\frac{4^{1/4}}{3^{1/4}} tr[\gamma_1\lambda_a^5] \ , \\
M_{a(5)}^{2,\text{origin}} &= \frac{i}{\sqrt{48\pi^3\mathcal{V}_2}} 3^{1/4} tr[\gamma_2\lambda_a^5] \ , \\
M_{a(5)}^{3,\text{origin}} &= \frac{i\sqrt{2\mathcal{V}_3}}{\sqrt{24\pi^3}} 16^{1/4} tr[\gamma_3\lambda_a^5] \ .
\end{aligned} \tag{C.25}
$$

The coefficients of C_Is are proportional to the coefficients of M_Is (without the traces):

$$
C^{k=1}_{(99,55)} = -4M^1_{(9,5)} \ , \quad C^{k=2}_{(99,55)} = -4M^2_{(9,5)} \ , \quad C^{k=3}_{(99,55)} = -M^3_{(9,5)} \ . \tag{C.26}
$$

for the various sectors in (C.24,C.25). In addition, for axionic exchange between D9-D9 and D5-D5 we have $\eta_1 = -\eta_5 = -1$, $\eta_2 = \eta_4 = 0$ however, between D9-D5 $\eta_1 = \eta_2 = -\eta_4 = -\eta_5 = 1$. In all cases $\eta_3 = 0$. Inserting the above to (C.13), we evaluate the symmetric tensor t_{ijl} and find agreement with the anomaly matrix (C.19).

Similarly, we evaluate the antisymmetric tensor (C.14) for Z_6':

$$
E_{ij} = E_{ijj} = -E_{jij} = \begin{pmatrix}
0 & -16 & 0 & -16 & 0 & 32 \\
16 & 0 & 0 & 0 & 16 & -32 \\
64 & -64 & 0 & -32 & 32 & 0 \\
-16 & 0 & 32 & 0 & -16 & 0 \\
0 & 16 & -32 & 16 & 0 & 0 \\
-32 & 32 & 0 & 64 & -64 & 0
\end{pmatrix} \ . \tag{C.27}
$$

Therefore, in the natural regularization scheme which treats democratically the anomalous currents, we need GCS for the Z_6' as well to cancel the anomalies.

We expect similar couplings to be present in other type of orientifold models, where anomaly cancellation is taken care by untwisted axions, like some intersecting / magnetized brane models [37, 38].

D. Computation of anomaly diagrams

In the following we define: $t^{ijk} = \sum_f [Q^i_f Q^j_f Q^k_f]$. The triangle amplitude in (6.1), in momentum space, is given by

$$
\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = i^3 t^{ijk} \int \frac{d^4p}{(2\pi)^4} \frac{Tr[\gamma_\mu(\not{p}+\not{k}_2)\gamma_\rho\not{p}\gamma_\nu(\not{p}-\not{k}_1)\gamma_5]}{(p+k_2)^2(p-k_1)^2p^2} \tag{D.1}
$$

and can be decomposed according to

$$
\begin{aligned}
\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = t^{ijk}[&A_1(k_1,k_2)\epsilon_{\mu\nu\rho\sigma}k_2^\sigma + A_2(k_1,k_2)\epsilon_{\mu\nu\rho\sigma}k_1^\sigma + B_1(k_1,k_2)k_{2\nu}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau \\
&+ B_2(k_1,k_2)k_{1\nu}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau + B_3(k_1,k_2)k_{2\rho}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau + B_4(k_1,k_2)k_{1\rho}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau]
\end{aligned} \tag{D.2}
$$

where A's and B's functions of k_1, k_2. In addition to the triangle diagram in (6.1), we have to add a similar triangle diagram with the exchange of $\{k_2, \rho\} \Leftrightarrow \{k_1, \nu\}$. The extra diagram will be similar to the above and the total result will be twice (D.2) if $A_1(k_1, k_2) = -A_2(k_2, k_1)$, $B_1(k_1, k_2) = -B_4(k_2, k_1)$, $B_2(k_1, k_2) = -B_3(k_2, k_1)$.

To evaluate the coefficient functions $A_1, A_2, B_1, B_2, B_3, B_4$, we use Feynman parametrization

$$\begin{aligned}\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} &= i^3 t^{ijk} \int d\alpha d\beta d\gamma\ \delta(1-\alpha-\beta-\gamma) \int \frac{d^4p}{(2\pi)^4} \frac{N_{\mu\nu\rho}(p,k_1,k_2)}{[\alpha(p+k_2)^2 + \beta(p-k_1)^2 + \gamma p^2]^3} \\ &= i^3 t^{ijk} \int d\alpha d\beta \int \frac{d^4p}{(2\pi)^4} \frac{N_{\mu\nu\rho}(p,k_1,k_2)}{[\alpha(p+k_2)^2 + \beta(p-k_1)^2 + (1-\alpha-\beta)p^2]^3} \ . \end{aligned} \tag{D.3}$$

We then make the change of variables $\tilde{p} = p + \alpha k_2 - \beta k_1$ and redefine back $\tilde{p} \to p$. We thus get

$$\begin{aligned}\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} &= i^3 t^{ijk} \int d\alpha d\beta \int \frac{d^4p}{(2\pi)^4} \frac{N_{\mu\nu\rho}(p,k_1,k_2)}{[p^2 + \alpha(1-\alpha)k_2^2 + \beta(1-\beta)k_1^2 + 2\alpha\beta k_1 k_2]^3} \\ &= i^3 t^{ijk} \int d\alpha d\beta \int \frac{d^4p}{(2\pi)^4} \frac{N_{\mu\nu\rho}(p,k_1,k_2)}{[p^2 - P^2 + \alpha k_2^2 + \beta k_1^2]^3} \ , \end{aligned} \tag{D.4}$$

where $P = \alpha k_2 - \beta k_1$. After the change of variables, the numerator is :

$$\begin{aligned} N_{\mu\nu\rho} &= Tr[\gamma_\mu(\not{p} - \not{P} + \not{k}_2)\gamma_\rho(\not{p} - \not{P})\gamma_\nu(\not{p} - \not{P} - \not{k}_1)\gamma_5] \\ &= -Tr[\gamma_\mu \not{p}\gamma_\rho \not{p}\gamma_\nu(\not{P} + \not{k}_1)\gamma_5] - Tr[\gamma_\mu(\not{P} - \not{k}_2)\gamma_\rho \not{p}\gamma_\nu \not{p}\gamma_5] - Tr[\gamma_\mu \not{p}\gamma_\rho \not{P}\gamma_\nu \not{p}\gamma_5] \\ &\quad - Tr[\gamma_\mu(\not{P} - \not{k}_2)\gamma_\rho \not{P}\gamma_\nu(\not{P} + \not{k}_1)\gamma_5] + \cdots \end{aligned} \tag{D.5}$$

We keep only terms with an even number of p's, which are not identically zero. Terms with two p's include a logarithmic divergence, whereas the last term in (D.5) is convergent. In dimensional regularization, using the fact that $p^\mu p^\nu \to g^{\mu\nu} p^2/d$, and $\gamma^\lambda \gamma_\mu \gamma_\lambda = -(d-2)\gamma_\mu$, $\gamma^\lambda \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\lambda = -2\gamma_\rho \gamma_\nu \gamma_\mu + (4-d)\gamma_\mu \gamma_\nu \gamma_\rho$, the p^2 terms can be written as

$$\begin{aligned} & p^2 \Big[\frac{2-d}{d}\ (P^\lambda + k_1^\lambda + P^\lambda - k_2^\lambda) + \frac{(d-6)}{d}\ P^\lambda\Big]\ Tr[\gamma_\mu \gamma_\rho \gamma_\nu \gamma_\lambda \gamma_5] \\ &= -\frac{p^2}{d}\ \Big(-[2-d+(d+2)\alpha]k_2^\lambda + [2-d+(d+2)\beta]k_1^\lambda\Big)\ (-4i\epsilon_{\mu\nu\rho\lambda}) \ . \end{aligned} \tag{D.6}$$

After integration these yield the functions A_1 and A_2 in (D.2), which are logarithmically divergent.

The term in the last line in (D.5) needs no regularization and can therefore be computed directly in four dimensions $d = 4$. The results is

$$\begin{aligned} & [(1-\alpha)P^2 - \beta k_1^2]\ k_2^\lambda\ Tr[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\lambda \gamma_5] - [(1-\beta)P^2 - \alpha k_2^2]\ k_1^\sigma\ Tr[\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_5] \\ & + [2\alpha(\alpha-1)k_{2\rho} - 2\alpha\beta k_{1\rho}]\ k_2^\lambda k_1^\sigma\ Tr[\gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma \gamma_5] \\ & + [2\alpha\beta k_{2\nu} - 2\beta(\beta-1)k_{1\nu}]\ k_2^\lambda k_1^\sigma\ Tr[\gamma_\mu \gamma_\rho \gamma_\lambda \gamma_\sigma \gamma_5] \end{aligned}$$

Now we can perform the integrals on p :

$$\int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2+M^2)^3} = \frac{1}{32\pi^2 M^2} \ , \tag{D.7}$$

whereas

$$\int \frac{d^4p}{(2\pi)^4} \frac{p^2}{(p^2+M^2)^3} \tag{D.8}$$

is logarithmically divergent and thus scheme dependent. On the contrary the finite coefficients B's of (D.2) can be determined unambiguously and read

$$\begin{aligned}
B_1(k_1,k_2) &= -\frac{i}{8\pi^2}\int_0^1 d\alpha d\beta \frac{2\alpha\beta}{\alpha k_2^2+\beta k_1^2-(\alpha k_2-\beta k_1)^2}\ , \\
B_2(k_1,k_2) &= -\frac{i}{8\pi^2}\int_0^1 d\alpha d\beta \frac{2\beta(1-\beta)}{\alpha k_2^2+\beta k_1^2-(\alpha k_2-\beta k_1)^2}\ , \\
B_3(k_1,k_2) &= -\frac{i}{8\pi^2}\int_0^1 d\alpha d\beta \frac{-2\alpha(1-\alpha)}{\alpha k_2^2+\beta k_1^2-(\alpha k_2-\beta k_1)^2}\ , \\
B_4(k_1,k_2) &= -\frac{i}{8\pi^2}\int_0^1 d\alpha d\beta \frac{-2\alpha\beta}{\alpha k_2^2+\beta k_1^2-(\alpha k_2-\beta k_1)^2}\ .
\end{aligned} \tag{D.9}$$

Notice that: $B_1(k_1,k_2) = -B_4(k_2,k_1) = -B_4(k_1,k_2)$, $B_2(k_1,k_2) = -B_3(k_2,k_1)$. The functions A_i, *a priori* logarithmically divergent, will be determined by imposing renormalization conditions on the three-point function.

D.1 All diagrams

Consider the three-point function of (abelian) gauge-bosons $\langle A^i_\mu(k_3)A^j_\nu(k_1)A^k_\rho(k_2)\rangle$ in momentum space. There are three contributions to this three-point function, linear in the momenta. One comes from the the irreducible CS-like vertex and gives a contribution

$$\langle A^i_\mu A^j_\nu A^k_\rho\rangle_{GCS} \ = \ -\epsilon^{\mu\nu\rho}{}_\sigma \left[E_{ijk}\ k_2^\sigma + E_{kij}\ k_1^\sigma + E_{jki}\ k_3^\sigma\right]\ , \tag{D.10}$$

where we have used antisymmetry in the first two indices.

The second contribution comes from axion-vector mixing terms and PQ couplings of the axions to $F-\tilde{F}$. We have the vertex

$$a^I(k_3)A^i_\mu(k_1)A^j_\mu(k_2) \to 2i\ C^I_{ij}\ \epsilon^{\mu\nu}{}_{\rho\sigma}\ k_2^\rho k_3^\sigma \tag{D.11}$$

where there is a factor of two coming from each field strength and a factor of 1/2 from the definition of the dual. The mixing term is

$$a^I(k)A^i_\mu(k) \to -M^i_I\ k^\mu \tag{D.12}$$

and the axion propagator

$$a^I(k)a^J(k) \to \frac{iG^{IJ}}{k^2}\ , \tag{D.13}$$

where axion indices are raised and lowered with the axion metric, to be taken to be canonical $G_{IJ} = \delta_{IJ}$ in what follows. Performing all six contractions we obtain

$$\langle A^i_\mu A^j_\nu A^k_\rho\rangle_{\rm contact} = -\Bigg[C^{ij}_I M^k_I \epsilon^{\mu\nu}{}_{\sigma\sigma'}\ k_3^\sigma k_1^{\sigma'}\ \frac{k_2^\rho}{k_2^2} + C^{ik}_I M^j_I\ \epsilon^{\mu\rho}{}_{\sigma\sigma'}\ k_3^\sigma k_2^{\sigma'}\ \frac{k_1^\nu}{k_1^2} \tag{D.14}$$

$$+C_I^{jk} M_I^i \epsilon^{\nu\rho}{}_{\sigma\sigma'}\ k_1^\sigma k_2^{\sigma'}\ \frac{k_3^\mu}{k_3^2}\Big] \ .$$

Combining all the contributions one gets

$$\Gamma^{ijk}_{\mu\nu\rho} = \Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} + \Gamma^{ijk}_{\mu\nu\rho}|_{axion} + \Gamma^{ijk}_{\mu\nu\rho}|_{CS} \ , \tag{D.15}$$

where

$$\begin{aligned}
\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} &= t^{ijk}[A_1\epsilon_{\mu\nu\rho\sigma}k_2^\sigma + A_2\epsilon_{\mu\nu\rho\sigma}k_1^\sigma + B_1 k_{2\nu}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau + B_2 k_{1\nu}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau \\
&\quad + B_3 k_{2\rho}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau + B_4 k_{1\rho}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau] \ , \qquad (D.16)\\
\Gamma^{ijk}_{\mu\nu\rho}|_{axion} &= -M_I^i C_I^{jk}\left(\frac{k_{3\mu}}{k_3^2}\right)\epsilon_{\nu\rho\sigma\tau}k_2^\sigma k_1^\tau - M_I^j C_I^{ki}\left(\frac{k_{1\nu}}{k_1^2}\right)\epsilon_{\rho\mu\tau\sigma}k_2^\sigma k_3^\tau - M_I^k C_I^{ij}\left(\frac{k_{2\rho}}{k_2^2}\right)\epsilon_{\mu\nu\tau\sigma}k_3^\sigma k_1^\tau \\
&= -M_I^i C_I^{jk}\frac{-(k_{1\mu}+k_{2\mu})}{(k_1+k_2)^2}\epsilon_{\nu\rho\sigma\tau}k_2^\sigma k_1^\tau + M_I^j C_I^{ki}\frac{k_{1\nu}}{k_1^2}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau - M_I^k C_I^{ij}\frac{k_{2\rho}}{k_2^2}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau \\
&\qquad (D.17)\\
\Gamma^{ijk}_{\mu\nu\rho}|_{CS} &= -E^{ij,k}\epsilon_{\mu\nu\rho\sigma}k_2^\sigma - E^{jk,i}\epsilon_{\nu\rho\mu\sigma}k_3^\sigma - E^{ki,j}\epsilon_{\rho\mu\nu\sigma}k_1^\sigma \\
&= -(E^{ij,k}-E^{jk,i})\epsilon_{\mu\nu\rho\sigma}k_2^\sigma - (E^{ki,j}-E^{jk,i})\epsilon_{\mu\nu\rho\sigma}k_1^\sigma \ . \qquad (D.18)
\end{aligned}$$

Imposing total Bose symmetry of the amplitude[12], the appropriate anomaly conditions for the triangle diagrams turn out to be

$$\left.\begin{array}{l}
k_1^\nu \Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = -t^{ijk}\frac{C_A}{3}\epsilon_{\mu\rho\sigma\tau}k_2^\sigma k_1^\tau \\
k_2^\rho \Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = t^{ijk}\frac{C_A}{3}\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau \\
-(k_1^\mu + k_2^\mu)\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = t^{ijk}\frac{C_A}{3}\epsilon_{\nu\rho\sigma\tau}k_2^\sigma k_1^\tau
\end{array}\right) \longrightarrow \begin{array}{l}
A_1 + k_1\cdot k_2 B_1 + k_1^2 B_2 = -\frac{C_A}{3} \\
A_2 + k_2^2 B_3 + k_1\cdot k_2 B_4 = \frac{C_A}{3} \\
A_1 - A_2 = \frac{C_A}{3}
\end{array}$$

where C_A is the standard coefficient of the axial anomaly. In this scheme, we can express the ambiguous amplitudes $A_1,\ A_2$ (a priori logarithmically divergent) in terms of the finite $B_1,\ B_2,\ B_3,\ B_4$ given in (D.9)[13].

Requiring the vanishing of the total anomaly we then find

$$\left.\begin{array}{l}
k_1^\nu \Gamma^{ijk}_{\mu\nu\rho} = 0 \\
k_2^\rho \Gamma^{ijk}_{\mu\nu\rho} = 0 \\
(k_1^\mu + k_2^\mu)\Gamma^{ijk}_{\mu\nu\rho} = 0
\end{array}\right) \longrightarrow \begin{array}{r}
t^{ijk}(A_1 + k_1\cdot k_2 B_1 + k_1^2 B_2) + M_I^j C_I^{ki} - E^{ij.k} + E^{jk.i} = 0 \\
t^{ijk}(A_2 + k_2^2 B_3 + k_1\cdot k_2 B_4) - M_I^k C_I^{ij} - E^{ki.j} + E^{jk.i} = 0 \\
t^{ijk}(A_1 - A_2) - M_I^i C_I^{jk} + E^{ki.j} - E^{ij.k} = 0
\end{array} \tag{D.20}$$

---

[12] In order to check the symmetry of the amplitude at the interchange of the external gauge bosons, we can use the identity:

$$\begin{aligned}
(k_{1\mu}+k_{2\mu})\epsilon_{\nu\rho\sigma\tau}k_2^\sigma k_1^\tau &= -(k_{2\nu}+k_{1\nu})\epsilon_{\rho\mu\sigma\tau}k_2^\sigma k_1^\tau - (k_{1\rho}+k_{2\rho})\epsilon_{\mu\nu\sigma\tau}k_2^\sigma k_1^\tau \\
&\quad + \epsilon_{\mu\nu\rho\sigma}[k_2^2 k_1^\sigma - k_1^2 k_2^\sigma - k_1.k_2(k_2^\sigma - k_1^\sigma)] \ . \qquad (D.19)
\end{aligned}$$

[13] We can change the scheme by redefining the scheme-dependent coeff. A_i. For example, if we require $k_2^\rho \Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = 0,\ k_1^\nu \Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = 0,\ (k_1^\mu + k_2^\mu)\Gamma^{ijk}_{\mu\nu\rho}|_{1-loop} = t^{ijk}C_A$, by choosing the anomaly to be contained in the third current, this can be done by the redefinitions $(A_1, A_2) \rightarrow (A_1 - C_A/3, A_2 + C_A/3)$.

We can also express the anomaly cancellation conditions in the form

$$
\begin{aligned}
t^{ijk}C_A &= M_I^k C_I^{ij} + M_I^j C_I^{ki} + M_I^i C_I^{jk} \ , \\
E^{ij.k} &= \frac{1}{3}(M_I^i C_I^{jk} - M_I^j C_I^{ki}) \ , \\
E^{jk.i} &= \frac{1}{3}(M_I^j C_I^{ki} - M_I^k C_I^{ij}) \ , \\
E^{ki.j} &= \frac{1}{3}(M_I^k C_I^{ij} - M_I^i C_I^{jk}) \ . \qquad \text{(D.21)}
\end{aligned}
$$

From the above equations it is easy to notice that for a gauge invariant model, the totaly symmetric part of $M_I^{(i}C_I^{jk)}$ cancels the anomaly (which is a totaly symmetric tensor C_At^{ijk}) and the antisymmetric part $M_I^{[i}C_I^{j]k}$ cancels $E^{[ij]k}$. Therefore, one can construct a gauge invariant model with anomalous fermion content and axions (without the GCS terms), one can construct a gauge invariant model without chiral fermion content ($t^{ijk}=0$) [23] but we cannot construct a model with anomalous fermions and GCS terms without axions.

References

[1] C. Bouchiat, J. Iliopoulos and P. Meyer, *"An Anomaly Free Version Of Weinberg's Model,"* Phys. Lett. B **38** (1972) 519.

[2] M. B. Green and J. H. Schwarz, *"Anomaly Cancellation In Supersymmetric D=10 Gauge Theory And Superstring Theory,"* Phys. Lett. B **149** (1984) 117 and Phys. Lett. B **151** (1985) 21.

[3] see, e.g. M. B. Green, J. H. Schwarz and E. Witten, "Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology", SPIRES entry Cambridge Univ. Press, 1987.

[4] M. Dine, N. Seiberg and E. Witten, *"Fayet-Iliopoulos Terms In String Theory,"* Nucl. Phys. B **289** (1987) 589.

[5] R. Blumenhagen, G. Honecker and T. Weigand, *"Loop-corrected compactifications of the heterotic string with line bundles,"* JHEP **0506** (2005) 020 [ArXiv:hep-th/0504232]; *"Supersymmetric (non-)abelian bundles in the type I and SO(32) heterotic string,"* JHEP **0508** (2005) 009 [ArXiv:hep-th/0507041].

[6] A. Sagnotti, *"Open Strings And Their Symmetry Groups,"* arXiv:hep-th/0208020; G. Pradisi and A. Sagnotti, *"Open String Orbifolds,"* Phys. Lett. B **216** (1989) 59; M. Bianchi, G. Pradisi and A. Sagnotti, *"Toroidal compactification and symmetry breaking in open string theories,"* Nucl. Phys. B **376** (1992) 365; For a review see C. Angelantonj and A. Sagnotti, *"Open strings,"* Phys. Rept. **371** (2002) 1 [Erratum-ibid. **376** (2003) 339] [ArXiv:hep-th/0204089] and references therein.

[7] E. G. Gimon and J. Polchinski, *"Consistency Conditions for Orientifolds and D-Manifolds,"* Phys. Rev. D **54** (1996) 1667 [ArXiv:hep-th/9601038]; A. Dabholkar and J. Park, *"Strings on Orientifolds,"* Nucl. Phys. B **477** (1996) 701 [ArXiv:hep-th/9604178]; E. G. Gimon and C. V. Johnson, *"K3 Orientifolds,"* Nucl. Phys. B **477** (1996) 715 [ArXiv:hep-th/9604129]; M. Berkooz and R. G. Leigh, *"A $D=4$ $N=1$ orbifold of type I strings,"* Nucl. Phys. B **483**, 187 (1997) [ArXiv:hep-th/9605049]; C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y. S. Stanev, *"Comments on Gepner models and type I vacua in string theory,"* Phys. Lett. B

387, 743 (1996) [ArXiv:hep-th/9607229]; C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y. S. Stanev, *"Chiral asymmetry in four-dimensional open- string vacua,"* Phys. Lett. B **385**, 96 (1996) [ArXiv:hep-th/9606169]; Z. Kakushadze and G. Shiu, *"A chiral $N = 1$ type I vacuum in four dimensions and its heterotic dual,"* Phys. Rev. D **56**, 3686 (1997) [ArXiv:hep-th/9705163]; M. Bianchi, *"A note on toroidal compactifications of the type I superstring and other superstring vacuum configurations with 16 supercharges,"* Nucl. Phys. B **528**, 73 (1998) [ArXiv:hep-th/9711201]; G. Aldazabal, A. Font, L. E. Ibanez and G. Violero, *"$D = 4$, $N = 1$, type IIB orientifolds,"* Nucl. Phys. B **536** (1998) 29 [ArXiv:hep-th/9804026].

[8] M. Bianchi and A. Sagnotti, *"On The Systematics Of Open String Theories,"* Phys. Lett. B **247** (1990) 517 and Nucl. Phys. B **361** (1991) 519.

[9] A. Sagnotti, *"A Note on the Green-Schwarz mechanism in open string theories,"* Phys. Lett. B **294** (1992) 196 [ArXiv:hep-th/9210127].

[10] L. E. Ibanez, R. Rabadan and A. M. Uranga, *"Anomalous U(1)'s in type I and type IIB $D = 4$, $N = 1$ string vacua,"* Nucl. Phys. B **542** (1999) 112 [ArXiv:hep-th/9808139].

[11] I. Antoniadis, C. Bachas and E. Dudas, *"Gauge couplings in four-dimensional type I string orbifolds,"* Nucl. Phys. B **560** (1999) 93 [ArXiv:hep-th/9906039].

[12] M. Bianchi and J. F. Morales, *"Anomalies and tadpoles,"* JHEP **0003** (2000) 030 [ArXiv:hep-th/0002149].

[13] E. Kiritsis, *"D-branes in standard model building, gravity and cosmology,"* Fortsch. Phys. **52** (2004) 200 [Phys. Rept. **421** (2005) 105] [ArXiv:hep-th/0310001].

[14] I. Antoniadis, E. Kiritsis and J. Rizos, *"Anomalous U(1)s in type I superstring vacua,"* Nucl. Phys. B **637** (2002) 92 [ArXiv:hep-th/0204153];
P. Anastasopoulos, *"4D anomalous U(1)'s, their masses and their relation to 6D anomalies,"* JHEP **0308** (2003) 005 [ArXiv:hep-th/0306042]; *"Anomalous U(1)s masses in non-supersymmetric open string vacua,"* Phys. Lett. B **588** (2004) 119 [ArXiv:hep-th/0402105].

[15] M. Cvetic and P. Langacker, *"Implications of Abelian Extended Gauge Structures From String Models,"* Phys. Rev. D **54** (1996) 3570 [ArXiv:hep-ph/9511378];
M. Cvetic and P. Langacker, *"Z' physics and supersymmetry,"* [ArXiv:hep-ph/9707451].

[16] I. Antoniadis, E. Kiritsis and T. N. Tomaras, *"A D-brane alternative to unification,"* Phys. Lett. B **486** (2000) 186 [ArXiv:hep-ph/0004214];
"D-brane Standard Model," Fortsch. Phys. **49** (2001) 573 [ArXiv:hep-th/0111269].
I. Antoniadis, E. Kiritsis, J. Rizos and T. N. Tomaras, *"D-branes and the standard model,"* Nucl. Phys. B **660** (2003) 81 [ArXiv:hep-th/0210263].

[17] E. Kiritsis and P. Anastasopoulos, *"The anomalous magnetic moment of the muon in the D-brane realization of the standard model,"* JHEP **0205** (2002) 054 [ArXiv:hep-ph/0201295].

[18] D. M. Ghilencea, L. E. Ibanez, N. Irges and F. Quevedo, *"TeV-scale Z' bosons from D-branes,"* JHEP **0208** (2002) 016 [ArXiv:hep-ph/0205083].

[19] B. Kors and P. Nath, "A Stückelberg extension of the standard model", *Phys. Lett. B* **586** (2004) p. 366, [ArXiv:hep-ph/0402047]
"A supersymmetric Stueckelberg U(1) extension of the MSSM," JHEP **0412** (2004) 005 [ArXiv:hep-ph/0406167];
"Aspects of the Stückelberg extension", *hep-ph/0503208*;

D. Feldman, Z. Liu and P. Nath, *"Probing a very narrow Z' boson with CDF and D0 data,"* [ArXiv:hep-ph/0603039].

[20] C. Coriano, N. Irges and E. Kiritsis, *"On the effective theory of low scale orientifold string vacua,"* [ArXiv:hep-ph/0510332].

[21] B. de Wit, P. G. Lauwers and A. Van Proeyen, *"Lagrangians Of N=2 Supergravity - Matter Systems,"* Nucl. Phys. B **255** (1985) 569.

[22] V. A. Tsokur and Y. M. Zinovev, *"Spontaneous supersymmetry breaking in $N = 4$ supergravity with matter,"* Phys. Atom. Nucl. **59** (1996) 2192 [Yad. Fiz. **59N12** (1996) 2277] [ArXiv:hep-th/9411104].

[23] L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, *"Gauging of flat groups in four dimensional supergravity,"* JHEP **0207** (2002) 010 [ArXiv:hep-th/0203206];
L. Andrianopoli, S. Ferrara and M. A. Lledo, *"Axion gauge symmetries and generalized Chern-Simons terms in $N = 1$ supersymmetric theories,"* JHEP **0404** (2004) 005 [ArXiv:hep-th/0402142].

[24] B. de Wit, H. Samtleben and M. Trigiante, *"On Lagrangians and gaugings of maximal supergravities,"* Nucl. Phys. B **655** (2003) 93 [ArXiv:hep-th/0212239].

[25] E. Dudas, A. Falkowski and S. Pokorski, *"Deconstructed U(1) and supersymmetry breaking,"* Phys. Lett. B **568** (2003) 281 [ArXiv:hep-th/0303155];
E. Dudas, T. Gherghetta and S. Groot Nibbelink, *"Vector / tensor duality in the five dimensional supersymmetric Green-Schwarz mechanism,"* Phys. Rev. D **70** (2004) 086012 [ArXiv:hep-th/0404094].

[26] C. A. Scrucca and M. Serone, *"Anomalies in field theories with extra dimensions,"* Int. J. Mod. Phys. A **19**, 2579 (2004) [ArXiv:hep-th/0403163];
C. A. Scrucca, M. Serone, L. Silvestrini and F. Zwirner, *"Anomalies in orbifold field theories,"* Phys. Lett. B **525**, 169 (2002) [ArXiv:hep-th/0110073];
C. T. Hill, *"Anomalies, Chern-Simons terms and chiral delocalization in extra dimensions,"* [ArXiv:hep-th/0601154].

[27] L. Alvarez-Gaume, *"An Introduction To Anomalies,"* HUTP-85/A092 SPIRES entry *Lectures given at Int. School on Mathematical Physics, Erice, Italy, Jul 1-14, 1985.*

[28] S. Weinberg, "The quantum theory of fields. Vol. 2: Modern applications", SPIRES entry Cambridge Univ. Press, 1996.

[29] J. Goldstone and F. Wilczek, *"Fractional Quantum Numbers On Solitons,"* Phys. Rev. Lett. **47** (1981) 986;
E. Witten, *"Global Aspects Of Current Algebra,"* Nucl. Phys. B **223** (1983) 422.

[30] E. D'Hoker and E. Farhi, *"Decoupling A Fermion In The Standard Electroweak Theory,"* Nucl. Phys. B **248** (1984) 77; *"Decoupling A Fermion Whose Mass Is Generated By A Yukawa Coupling: The General Case,"* Nucl. Phys. B **248** (1984) 59.

[31] D. Z. Freedman and B. Kors, *"Kaehler anomalies, Fayet-Iliopoulos couplings, and flux vacua,"* [ArXiv:hep-th/0509217].

[32] T. Inami, H. Kanno and T. Kubota, *"Hexagon Gauge Anomaly Of The Type I Superstring In The Path Integral Method,"* Phys. Lett. B **199** (1987) 389.

[33] J. Polchinski and Y. Cai, *"Consistency Of Open Superstring Theories,"* Nucl. Phys. B **296**, 91 (1988).

[34] J. F. Morales, C. A. Scrucca and M. Serone, *"Anomalous couplings for D-branes and O-planes,"* Nucl. Phys. B **552**, 291 (1999) [ArXiv:hep-th/9812071];
C. A. Scrucca and M. Serone, *"Anomalies and inflow on D-branes and O-planes,"* Nucl. Phys. B **556**, 197 (1999) [ArXiv:hep-th/9903145];
C. A. Scrucca and M. Serone, *"Gauge and gravitational anomalies in $D = 4$ $N = 1$ orientifolds,"* JHEP **9912**, 024 (1999) [ArXiv:hep-th/9912108];
M. Klein, *"Anomaly cancellation in $D = 4$, $N = 1$ orientifolds and linear/chiral multiplet duality,"* Nucl. Phys. B **569**, 362 (2000) [ArXiv:hep-th/9910143].

[35] I. Antoniadis, C. Bachas, C. Fabre, H. Partouche and T. R. Taylor, *"Aspects of type I - type II - heterotic triality in four dimensions,"* Nucl. Phys. B **489**, 160 (1997) [ArXiv:hep-th/9608012];
I. Antoniadis, H. Partouche and T. R. Taylor, *"Lectures on heterotic-type I duality,"* Nucl. Phys. Proc. Suppl. **61A**, 58 (1998) [Nucl. Phys. Proc. Suppl. **67**, 3 (1998)] [ArXiv:hep-th/9706211].

[36] S. R. Coleman and B. Grossman, *"'T Hooft's Consistency Condition As A Consequence Of Analyticity And Unitarity,"* Nucl. Phys. B **203** (1982) 205.

[37] M. Berkooz, M. R. Douglas and R. G. Leigh, *"Branes intersecting at angles,"* Nucl. Phys. B **480** (1996) 265 [ArXiv:hep-th/9606139];
R. Blumenhagen, L. Goerlich, B. Kors and D. Lust, *"Noncommutative compactifications of type I strings on tori with magnetic background flux,"* JHEP **0010** (2000) 006 [ArXiv:hep-th/0007024];
G. Aldazabal, S. Franco, L. E. Ibanez, R. Rabadan and A. M. Uranga, *"Intersecting brane worlds,"* JHEP **0102** (2001) 047 [ArXiv:hep-ph/0011132].

[38] C. Bachas, *"A Way to break supersymmetry,"* [ArXiv:hep-th/9503030];
M. Bianchi and Y. S. Stanev, *"Open strings on the Neveu-Schwarz pentabrane,"* Nucl. Phys. B **523**, 193 (1998) [ArXiv:hep-th/9711069];
C. Angelantonj, I. Antoniadis, E. Dudas and A. Sagnotti, *"Type-I strings on magnetised orbifolds and brane transmutation,"* Phys. Lett. B **489** (2000) 223 [ArXiv:hep-th/0007090] ;
C. Angelantonj and A. Sagnotti, *"Type-I vacua and brane transmutation,"* [ArXiv:hep-th/0010279];
M. Larosa and G. Pradisi, *"Magnetized four-dimensional $Z_2 \times Z_2$ orientifolds,"* Nucl. Phys. B **667**, 261 (2003) [ArXiv:hep-th/0305224];
I. Antoniadis and T. Maillard, *"Moduli stabilization from magnetic fluxes in type I string theory,"* Nucl. Phys. B **716**, 3 (2005) [ArXiv:hep-th/0412008];
I. Antoniadis, A. Kumar and T. Maillard, *"Moduli stabilization with open and closed string fluxes,"* [ArXiv:hep-th/0505260];
E. Dudas and C. Timirgaziu, *"Internal magnetic fields and supersymmetry in orientifolds,"* Nucl. Phys. B **716**, 65 (2005) [ArXiv:hep-th/0502085];
M. Bianchi and E. Trevigne, *"The open story of the magnetic fluxes,"* JHEP **0508**, 034 (2005) [ArXiv:hep-th/0502147] and *"Gauge thresholds in the presence of oblique magnetic fluxes,"* JHEP **0601**, 092 (2006) [ArXiv:hep-th/0506080].